

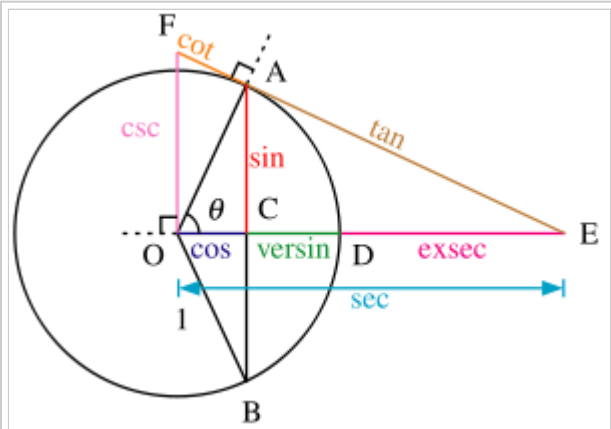
List of trigonometric identities

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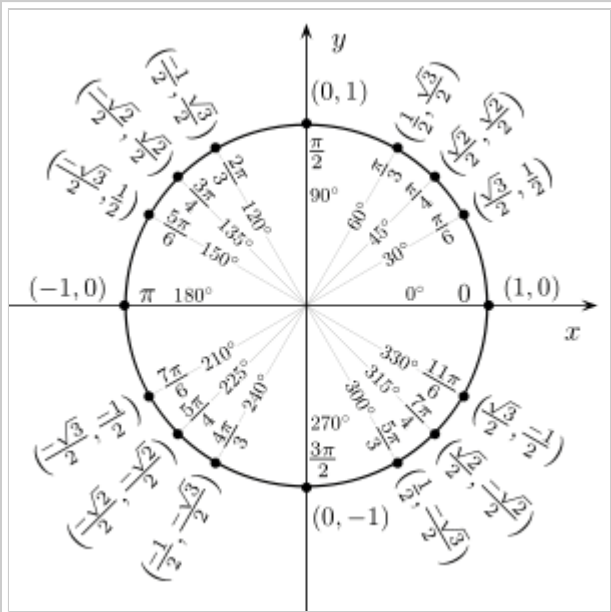
In mathematics, **trigonometric identities** are equalities involving trigonometric functions that are true for all values of the occurring variables. These identities are useful whenever expressions involving trigonometric functions need to be simplified. An important application is the integration of non-trigonometric functions: a common trick involves first using the substitution rule with a trigonometric function, and then simplifying the resulting integral with a trigonometric identity.

Contents

- 1 Notation
- 2 Definitions
- 3 Periodicity, symmetry, and shifts
 - 3.1 Periodicity
 - 3.2 Symmetry
 - 3.3 Shifts
 - 3.4 Linear combinations
- 4 Pythagorean identities
- 5 Angle sum and difference identities
- 6 Double-angle formula
- 7 Triple-angle formula
- 8 Multiple-angle formula
- 9 Power-reduction formulæ
- 10 Half-angle formula
- 11 Product-to-sum identities
- 12 Sum-to-product identities
- 13 Other sums of trigonometric functions
- 14 Inverse trigonometric functions
- 15 Trigonometric conversions
- 16 Exponential forms
- 17 Infinite product formulæ
- 18 The Gudermannian function
- 19 Identities without variables
- 20 Calculus
 - 20.1 Implications
- 21 Geometric proofs
 - 21.1 $\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)$
 - 21.2 $\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$
- 22 Proofs of $\cos(x - y)$ and $\sin(x - y)$ formulæ
 - 22.1 $\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y)$
 - 22.2 $\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$
- 23 See also
- 24 External links



All of the trigonometric functions of an angle θ can be constructed geometrically in terms of a unit circle centered at O .



The unit circle

Notation

The following notations hold for all six trigonometric functions: sine (\sin), cosine (\cos), tangent (\tan), cotangent (\cot), secant (\sec), and cosecant (\csc). For brevity, only the *sine* case is given in the table.

Notation	Reading	Description	Definition
$\sin^2(x)$	"sine squared [of] x "	the square of sine; sine to the second power	$\sin^2(x) = (\sin(x))^2$
$\arcsin(x)$	"arcsine [of] x "	the inverse function for sine	$\arcsin(x) = y$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$(\sin(x))^{-1}$	"sine [of] x , to the [power of] minus-one"	the reciprocal of sine; the multiplicative inverse of sine	$(\sin(x))^{-1} = 1 / \sin(x) = \csc(x)$
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$\arcsin(x)$ can also be written $\sin^{-1}(x)$; this must not be confused with $(\sin(x))^{-1}$.

Definitions

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$

$$\sec(x) = \frac{1}{\cos(x)} \quad \csc(x) = \frac{1}{\sin(x)}$$

For more information, including definitions based on the sides of a right triangle, see trigonometric function.

Periodicity, symmetry, and shifts

These are most easily shown from the unit circle:

Periodicity

The sine, cosine, secant, and cosecant functions have period 2π (a full circle):

$$\begin{aligned}\sin(x) &= \sin(x + 2k\pi) \\ \cos(x) &= \cos(x + 2k\pi)\end{aligned}$$

$$\begin{aligned}\sec(x) &= \sec(x + 2k\pi) \\ \csc(x) &= \csc(x + 2k\pi)\end{aligned}$$

The tangent and cotangent functions have period π (a half-circle):

$$\begin{aligned}\tan(x) &= \tan(x + k\pi) \\ \cot(x) &= \cot(x + k\pi)\end{aligned}$$

Symmetry

The symmetries along $x \rightarrow -x$, $x \rightarrow \pi/2 - x$ and $x \rightarrow \pi - x$ for the trigonometric functions are:

$$\begin{array}{lll}\sin(-x) = -\sin(x), & \sin\left(\frac{\pi}{2} - x\right) = \cos(x), & \sin(\pi - x) = \sin(x) \\ \cos(-x) = \cos(x), & \cos\left(\frac{\pi}{2} - x\right) = \sin(x), & \cos(\pi - x) = -\cos(x) \\ \tan(-x) = -\tan(x), & \tan\left(\frac{\pi}{2} - x\right) = \cot(x), & \tan(\pi - x) = -\tan(x) \\ \cot(-x) = -\cot(x), & \cot\left(\frac{\pi}{2} - x\right) = \tan(x), & \cot(\pi - x) = -\cot(x) \\ \sec(-x) = \sec(x), & \sec\left(\frac{\pi}{2} - x\right) = \csc(x), & \sec(\pi - x) = -\sec(x) \\ \csc(-x) = -\csc(x), & \csc\left(\frac{\pi}{2} - x\right) = \sec(x), & \csc(\pi - x) = \csc(x)\end{array}$$

Shifts

Among the simplest shifts (other than shifts by the period of each of these periodic functions) are shifts by $\pi/2$ and π :

$$\begin{array}{ll} \sin\left(x + \frac{\pi}{2}\right) = \cos(x), & \sin(x + \pi) = -\sin(x) \\ \cos\left(x + \frac{\pi}{2}\right) = -\sin(x), & \cos(x + \pi) = -\cos(x) \\ \tan\left(x + \frac{\pi}{2}\right) = -\cot(x), & \tan(x + \pi) = \tan(x) \\ \cot\left(x + \frac{\pi}{2}\right) = -\tan(x), & \cot(x + \pi) = \cot(x) \\ \sec\left(x + \frac{\pi}{2}\right) = -\csc(x), & \sec(x + \pi) = -\sec(x) \\ \csc\left(x + \frac{\pi}{2}\right) = \sec(x), & \csc(x + \pi) = -\csc(x) \end{array}$$

Linear combinations

For some purposes it is important to know that any linear combination of sine waves of the same period but different phase shifts is also a sine wave with the same period, but a different phase shift. In other words, we have

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \cdot \sin(x + \varphi)$$

where

$$\varphi = \begin{cases} \arctan(b/a), & \text{if } a \geq 0; \\ \arctan(b/a) \pm \pi, & \text{if } a < 0. \end{cases}$$

Pythagorean identities

These identities are based on the Pythagorean theorem. The first is sometimes simply called the Pythagorean trigonometric identity.

$$\sin^2(x) + \cos^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$$1 + \cot^2(x) = \csc^2(x)$$

Note that the second equation is obtained from the first by dividing both sides by $\cos^2(x)$. To get the third equation, divide the first by $\sin^2(x)$ instead.

Angle sum and difference identities

These are also known as the *addition and subtraction theorems* or *formulae*. The quickest way to prove these is Euler's formula. The tangent formula follows from the other two. A geometric proof of the $\sin(x + y)$ identity is given at the end of this article.

$$\sin(x \pm y) = \sin(x) \cos(y) \pm \cos(x) \sin(y)$$

(When "+" is on the left side, then "+" is on the right, and vice versa.)

$$\cos(x \pm y) = \cos(x) \cos(y) \mp \sin(x) \sin(y)$$

(When "+" is on the left side, then "-" is on the right, and vice versa.)

$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x) \tan(y)}$$

$$\operatorname{cis}(x + y) = \operatorname{cis}(x) \operatorname{cis}(y)$$

$$\operatorname{cis}(x - y) = \frac{\operatorname{cis}(x)}{\operatorname{cis}(y)}$$

where

$$\operatorname{cis}(x) = \exp(ix) = e^{ix} = \cos(x) + i \sin(x)$$

and

$$i^2 = -1.$$

See also Ptolemaios' theorem.

Double-angle formula

These can be shown by substituting $x = y$ in the addition theorems, and using the Pythagorean formula. Or use de Moivre's formula with $n = 2$.

$$\sin(2x) = 2 \sin(x) \cos(x) = \frac{2 \tan(x)}{1 + \tan^2(x)}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x) = \frac{1 - \tan^2(x)}{1 + \tan^2(x)}$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

$$\cot(2x) = \frac{\cot(x) - \tan(x)}{2}$$

The double-angle formula can also be used to find Pythagorean triples. If (a, b, c) are the lengths of the sides of a right triangle, then $(a^2 - b^2, 2ab, c^2)$ also form a right triangle, where angle B is the angle being doubled. If $a^2 - b^2$ is negative, take its opposite and use the supplement of $2B$ in place of $2B$.

Triple-angle formula

$$\sin(3x) = 3 \sin(x) - 4 \sin^3(x)$$

$$\cos(3x) = 4 \cos^3(x) - 3 \cos(x)$$

$$\tan(3x) = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2(x)}$$

Multiple-angle formula

If T_n is the n th Chebyshev polynomial then

$$\cos(nx) = T_n(\cos(x)).$$

If S_n is the n th spread polynomial, then

$$\sin^2(n\theta) = S_n(\sin^2 \theta).$$

de Moivre's formula:

$$\cos(nx) + i \sin(nx) = (\cos(x) + i \sin(x))^n$$

The **Dirichlet kernel** $D_n(x)$ is the function occurring on both sides of the next identity:

$$1 + 2 \cos(x) + 2 \cos(2x) + 2 \cos(3x) + \cdots + 2 \cos(nx) = \frac{\sin\left(\left(n + \frac{1}{2}\right)x\right)}{\sin(x/2)}.$$

The convolution of any integrable function of period 2π with the Dirichlet kernel coincides with the function's n th-degree Fourier approximation. The same holds for any measure or generalized function.

Power-reduction formulæ

Solve the second and third versions of the cosine double-angle formula for $\cos^2(x)$ and $\sin^2(x)$, respectively.

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\sin^2(x) \cos^2(x) = \frac{1 - \cos(4x)}{8}$$

$$\sin^3(x) = \frac{3 \sin(x) - \sin(3x)}{4}$$

$$\cos^3(x) = \frac{3 \cos(x) + \cos(3x)}{4}$$

Half-angle formula

Sometimes the formulæ in the previous section are called *half-angle formulæ*. To see why, substitute $x/2$ for x in the power reduction formulæ, then solve for $\cos(x/2)$ and $\sin(x/2)$ to get:

$$\cos\left(\frac{x}{2}\right) = \pm \sqrt{\frac{1 + \cos(x)}{2}}$$

$$\sin\left(\frac{x}{2}\right) = \pm \sqrt{\frac{1 - \cos(x)}{2}}$$

These may also be called the *half-angle formulæ*. Then

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x/2)}{\cos(x/2)} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}. \quad (1)$$

Multiply both numerator and denominator inside the radical by $1 + \cos x$, then simplify (using a Pythagorean identity):

$$\begin{aligned} \tan\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{(1 - \cos x)(1 + \cos x)}{(1 + \cos x)(1 + \cos x)}} = \pm \sqrt{\frac{1 - \cos^2 x}{(1 + \cos x)^2}} \\ &= \frac{\sin x}{1 + \cos x}. \end{aligned}$$

Likewise, multiplying both numerator and denominator inside the radical — in equation (1) — by $1 - \cos x$, then simplifying:

$$\begin{aligned} \tan\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{(1 - \cos x)(1 - \cos x)}{(1 + \cos x)(1 - \cos x)}} = \pm \sqrt{\frac{(1 - \cos x)^2}{(1 - \cos^2 x)}} \\ &= \frac{1 - \cos x}{\sin x}. \end{aligned}$$

Thus, the pair of half-angle formulæ for the tangent are:

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1 + \cos(x)} = \frac{1 - \cos(x)}{\sin(x)}.$$

We also have

$$\tan\left(\frac{x}{2}\right) = \csc(x) - \cot(x),$$

$$\cot\left(\frac{x}{2}\right) = \csc(x) + \cot(x).$$

If we set

$$t = \tan\left(\frac{x}{2}\right),$$

then

$$\sin(x) = \frac{2t}{1 + t^2} \quad \text{and} \quad \cos(x) = \frac{1 - t^2}{1 + t^2} \quad \text{and} \quad e^{ix} = \frac{1 + it}{1 - it}.$$

This substitution of t for $\tan(x/2)$, with the consequent replacement of $\sin(x)$ by $2t/(1 + t^2)$ and $\cos(x)$ by $(1 - t^2)/(1 + t^2)$ is useful in calculus for converting rational functions in $\sin(x)$ and $\cos(x)$ to functions of t in order to find their antiderivatives. For more information see tangent half-angle formula.

Product-to-sum identities

These can be proven by expanding their right-hand sides using the angle addition theorems.

$$\cos(x) \cos(y) = \frac{\cos(x-y) + \cos(x+y)}{2}$$

$$\sin(x) \sin(y) = \frac{\cos(x-y) - \cos(x+y)}{2}$$

$$\sin(x) \cos(y) = \frac{\sin(x-y) + \sin(x+y)}{2}$$

Sum-to-product identities

Replace x by $(x+y)/2$ and y by $(x-y)/2$ in the product-to-sum formulæ.

$$\cos(x) + \cos(y) = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) + \sin(y) = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos(x) - \cos(y) = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

$$\sin(x) - \sin(y) = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

If x , y , and z are the three angles of any triangle, or in other words

if $x + y + z = \pi = \text{half circle}$,

then $\tan(x) + \tan(y) + \tan(z) = \tan(x) \tan(y) \tan(z)$,

and $\sin(2x) + \sin(2y) + \sin(2z) = 4 \sin(x) \sin(y) \sin(z)$.

(If any of x , y , z is a right angle, one should take both sides to be ∞ . This is neither $+\infty$ nor $-\infty$; for present purposes it makes sense to add just one point at infinity to the real line, that is approached by $\tan(\theta)$ as $\tan(\theta)$ either increases through positive values or decreases through negative values. This is a one-point compactification of the real line.)

Other sums of trigonometric functions

For any a and b :

$$a \cos(x) + b \sin(x) = \sqrt{a^2 + b^2} \cos(x - \arctan(b, a))$$

where $\arctan(y, x)$ is the generalization of $\arctan(y/x)$ which covers the entire circular range (see also the account of this same identity in "symmetry, periodicity, and shifts" above for this generalization of \arctan).

$$\tan(x) + \sec(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

The above identity is sometimes convenient to know when thinking about the Gudermanian function.

If x , y , and z are the three angles of any triangle, i.e. $x + y + z = \pi$ then,

$$\cot(x) \cot(y) + \cot(y) \cot(z) + \cot(z) \cot(x) = 1.$$

Inverse trigonometric functions

$$\arcsin(x) + \arccos(x) = \pi/2$$

$$\arctan(x) + \operatorname{arccot}(x) = \pi/2.$$

$$\arctan(x) + \arctan(1/x) = \begin{cases} \pi/2, & \text{if } x > 0 \\ -\pi/2, & \text{if } x < 0 \end{cases}$$

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right) + \begin{cases} \pi, & \text{if } x, y > 0 \\ -\pi, & \text{if } x, y < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\sin(\arccos(x)) = \sqrt{1-x^2}$$

$$\cos(\arcsin(x)) = \sqrt{1-x^2}$$

$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$$

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$$

$$\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$$

$$\tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$$

Trigonometric conversions

Every trigonometric function can be related directly to every other trigonometric function. Such relations can be expressed by means of inverse trigonometric functions as follows: let ϕ and ψ represent a pair of trigonometric functions, and let $\operatorname{arc}\psi$ be the inverse of ψ , such that $\psi(\operatorname{arc}\psi(x)) = x$. Then $\phi(\operatorname{arc}\psi(x))$ can be expressed as an algebraic formula in terms of x . Such formulæ are shown in the table below: ϕ can be made equal to the head of one of the rows, and ψ can be equated to the head of a column:

Table of conversion formulæ

$\phi \setminus \psi$	sin	cos	tan	csc	sec	cot
sin	x	$\sqrt{1-x^2}$	$\frac{x}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{\sqrt{x^2-1}}{x}$
cos	$\sqrt{1-x^2}$	x	$\frac{1}{\sqrt{1+x^2}}$	$\frac{\sqrt{x^2-1}}{x}$	$\frac{1}{x}$	$\frac{x}{\sqrt{1+x^2}}$

tan	$\frac{x}{\sqrt{1-x^2}}$	$\frac{\sqrt{1-x^2}}{x}$	x	$\frac{1}{\sqrt{x^2-1}}$	$\sqrt{x^2-1}$	$\frac{1}{x}$
csc	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{\sqrt{1+x^2}}{x}$	x	$\frac{x}{\sqrt{x^2-1}}$	$\sqrt{1+x^2}$
sec	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$\sqrt{1+x^2}$	$\frac{x}{\sqrt{x^2-1}}$	x	$\frac{\sqrt{1+x^2}}{x}$
cot	$\frac{\sqrt{1-x^2}}{x}$	$\frac{x}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$\sqrt{x^2-1}$	$\frac{1}{\sqrt{x^2-1}}$	x

One procedure that can be used to obtain the elements of this table is as follows:

Given trigonometric functions ϕ and ψ , what is $\phi(\arccsc(x))$ equal to?

1. Find an equation that relates $\phi(u)$ and $\psi(u)$ to each other:

$$f(\phi(u), \psi(u)) = 0$$

2. Let $u = \arccsc(x)$, so that:

$$f(\phi(\arccsc(x)), \psi(\arccsc(x))) = 0$$

$$f(\phi(\arccsc(x)), x) = 0$$

3. Solve the last equation for $\phi(\arccsc(x))$.

Example. What is $\cot(\arccsc(x))$ equal to? First, find an equation which relations the functions \cot and \csc to each other, such as

$$\cot^2 u + 1 = \csc^2 u.$$

Second, let $u = \arccsc(x)$:

$$\cot^2(\arccsc(x)) + 1 = \csc^2(\arccsc(x)),$$

$$\cot^2(\arccsc(x)) + 1 = x^2.$$

Third, solve this equation for $\cot(\arccsc(x))$:

$$\cot^2(\arccsc(x)) = x^2 - 1,$$

$$\cot(\arccsc(x)) = \pm\sqrt{x^2 - 1},$$

and this is the formula which shows up in the sixth row and fourth column of the table.

Exponential forms

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

where $i^2 = -1$.

Infinite product formulæ

For applications to special functions, the following infinite product formulæ for trigonometric functions are useful:

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2} \right)$$

$$\sinh x = x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 n^2} \right)$$

$$\cos x = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 (n - \frac{1}{2})^2} \right)$$

$$\cosh x = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 (n - \frac{1}{2})^2} \right)$$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos \left(\frac{x}{2^n} \right)$$

The Gudermannian function

The Gudermannian function relates the circular and hyperbolic trigonometric functions without resorting to complex numbers; see that article for details.

Identities without variables

Richard Feynman is reputed to have learned as a boy, and always remembered, the following curious identity:

$$\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = \frac{1}{8}.$$

However, this is a special case of an identity that contains one variable:

$$\prod_{j=0}^{k-1} \cos(2^j x) = \frac{\sin(2^k x)}{2^k \sin(x)}.$$

The following is perhaps not as readily generalized to an identity containing variables:

$$\cos 24^\circ + \cos 48^\circ + \cos 96^\circ + \cos 168^\circ = \frac{1}{2}.$$

Degree measure ceases to be more felicitous than radian measure when we consider this identity with 21 in the denominators:

$$\begin{aligned} &\cos \left(\frac{2\pi}{21} \right) + \cos \left(2 \cdot \frac{2\pi}{21} \right) + \cos \left(4 \cdot \frac{2\pi}{21} \right) \\ &\quad + \cos \left(5 \cdot \frac{2\pi}{21} \right) + \cos \left(8 \cdot \frac{2\pi}{21} \right) + \cos \left(10 \cdot \frac{2\pi}{21} \right) = \frac{1}{2}. \end{aligned}$$

The factors 1, 2, 4, 5, 8, 10 may start to make the pattern clear: they are those integers less than 21/2 that are relatively prime to (or have no prime factors in common with) 21. The last several examples are corollaries of a basic fact about the irreducible cyclotomic polynomials: the cosines are the real parts of the zeroes of those polynomials; the sum of the zeroes is the Möbius function evaluated at (in the very last case above) 21; only half of the zeroes are present above. The two identities preceding this last one arise in the same fashion with 21 replaced by 10 and 15, respectively.

An efficient way to compute π is based on the following identity without variables, due to Machin:

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$

or, alternatively, by using Euler's formula:

$$\frac{\pi}{4} = 5 \arctan \frac{1}{7} + 2 \arctan \frac{3}{79}.$$

$$\sin 0 = \sin 0^\circ = 0 = \cos 90^\circ = \cos \left(\frac{\pi}{2}\right)$$

$$\sin \left(\frac{\pi}{6}\right) = \sin 30^\circ = 1/2 = \cos 60^\circ = \cos \left(\frac{\pi}{3}\right)$$

$$\sin \left(\frac{\pi}{4}\right) = \sin 45^\circ = \sqrt{2}/2 = \cos 45^\circ = \cos \left(\frac{\pi}{4}\right)$$

$$\sin \left(\frac{\pi}{3}\right) = \sin 60^\circ = \sqrt{3}/2 = \cos 30^\circ = \cos \left(\frac{\pi}{6}\right)$$

$$\sin \left(\frac{\pi}{2}\right) = \sin 90^\circ = 1 = \cos 0^\circ = \cos 0$$

$$\sin \frac{\pi}{7} = \frac{\sqrt{7}}{6} - \frac{\sqrt{7}}{189} \sum_{j=0}^{\infty} \frac{(3j+1)!}{189^j j! (2j+2)!}$$

$$\sin \frac{\pi}{18} = \frac{1}{6} \sum_{j=0}^{\infty} \frac{(3j)!}{27^j j! (2j+1)!}$$

With the golden ratio φ :

$$\cos \left(\frac{\pi}{5}\right) = \cos 36^\circ = \frac{\sqrt{5}+1}{4} = \varphi/2$$

$$\sin \left(\frac{\pi}{10}\right) = \sin 18^\circ = \frac{\sqrt{5}-1}{4} = \frac{\varphi-1}{2} = \frac{1}{2\varphi}$$

Also see exact trigonometric constants.

Calculus

In calculus the relations stated below require angles to be measured in radians; the relations would become more complicated if angles were measured in another unit such as degrees. If the trigonometric functions are defined in terms of geometry, then their

derivatives can be found by verifying two limits. The first is:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1,$$

verified using the unit circle and squeeze theorem. It may be tempting to propose to use L'Hôpital's rule to establish this limit. However, if one uses this limit in order to prove that the derivative of the sine is the cosine, and then uses the fact that the derivative of the sine is the cosine in applying L'Hôpital's rule, one is reasoning circularly—a logical fallacy. The second limit is:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x} = 0,$$

verified using the identity $\tan(x/2) = (1 - \cos(x))/\sin(x)$. Having established these two limits, one can use the limit definition of the derivative and the addition theorems to show that $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$. If the sine and cosine functions are defined by their Taylor series, then the derivatives can be found by differentiating the power series term-by-term.

$$\frac{d}{dx} \sin(x) = \cos(x)$$

The rest of the trigonometric functions can be differentiated using the above identities and the rules of differentiation. We have:

$$\begin{array}{lll} \frac{d}{dx} \sin x = \cos x & , & \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \cos x = -\sin x & , & \frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \tan x = \sec^2 x & , & \frac{d}{dx} \arctan x = \frac{1}{1+x^2} \\ \frac{d}{dx} \cot x = -\csc^2 x & , & \frac{d}{dx} \operatorname{arccot} x = \frac{-1}{1+x^2} \\ \frac{d}{dx} \sec x = \tan x \sec x & , & \frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx} \csc x = -\csc x \cot x & , & \frac{d}{dx} \operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2-1}} \end{array}$$

The integral identities can be found in "list of integrals of trigonometric functions".

Implications

The fact that the differentiation of trigonometric functions (sine and cosine) results in linear combinations of the same two functions is of fundamental importance to many fields of mathematics, including differential equations and fourier transformations.

Geometric proofs

These proofs apply directly only to acute angles, but the truth of these identities in the case of acute angles can be used to infer their truth in more general cases.

$$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)$$

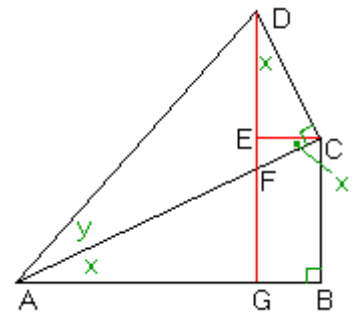
In the figure the angle x is part of right angled triangle ABC, and the angle y part of right angled triangle ACD. Then construct DG perpendicular to AB and construct CE parallel to AB.

Angle x = Angle BAC = Angle ACE = Angle CDE.

EG = BC.

$$\sin(x + y)$$

$$\begin{aligned}
&= \frac{DG}{AD} \\
&= \frac{EG + DE}{AD} \\
&= \frac{BC + DE}{AD} \\
&= \frac{BC}{AD} + \frac{DE}{AD} \\
&= \frac{BC}{AD} \cdot \frac{AC}{AC} + \frac{DE}{AD} \cdot \frac{CD}{CD} \\
&= \frac{BC}{AC} \cdot \frac{AC}{AD} + \frac{DE}{CD} \cdot \frac{CD}{AD} \\
&= \sin(x) \cos(y) + \cos(x) \sin(y).
\end{aligned}$$



$$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$$

Using the above figure:

$$\begin{aligned}
&\cos(x + y) \\
&= \frac{AG}{AD} \\
&= \frac{AB - GB}{AD} \\
&= \frac{AB - EC}{AD} \\
&= \frac{AB}{AD} - \frac{EC}{AD} \\
&= \frac{AB}{AD} \cdot \frac{AC}{AC} - \frac{EC}{AD} \cdot \frac{CD}{CD} \\
&= \frac{AB}{AC} \cdot \frac{AC}{AD} - \frac{EC}{CD} \cdot \frac{CD}{AD} \\
&= \cos(x) \cos(y) - \sin(x) \sin(y).
\end{aligned}$$

Proofs of $\cos(x - y)$ and $\sin(x - y)$ formulæ

The formulæ for $\cos(x - y)$ and $\sin(x - y)$ are easily proven using the formulæ for $\cos(x + y)$ and $\sin(x + y)$, respectively

$$\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y)$$

To begin, we substitute y with $-y$ into the $\sin(x + y)$ formula:

$$\sin(x + (-y)) = \sin(x) \cos(-y) + \cos(x) \sin(-y).$$

Using the fact that sine is an odd function and cosine is an even function, we get

$$\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y).$$

$$\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$$

To begin, we substitute y with $-y$ into the $\cos(x + y)$ formula:

$$\cos(x + (-y)) = \cos(x) \cos(-y) - \sin(x) \sin(-y).$$

Using the fact that sine is an odd function and cosine is an even function, we get

$$\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y).$$

See also

- Proofs of trigonometric identities
- Uses of trigonometry
- Tangent half-angle formula
- Law of cosines
- Law of sines
- Law of tangents
- Pythagorean theorem
- Exact trigonometric constants

External links

- A one page proof (<http://oregonstate.edu/~barnesc/documents/trigonometry.pdf>) of many trigonometric identities using Euler's formula, by Connelly Barnes.

Retrieved from "http://en.wikipedia.org/wiki/List_of_trigonometric_identities"

Categories: Mathematical identities | Trigonometry

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