Sets and Relations

SETS, ELEMENTS

Any well defined list or collection of objects is called a set; the objects comprising the set are called its *elements* or *members*. We write

$$p \in A \quad \text{if } p \text{ is an element in the set } A$$

If every element of A also belongs to a set B, i.e. if $x \in A$ implies $x \in B$, then A is called a *subset* of B or is said to be *contained* in B; this is denoted by

$$A \subset B \quad \text{or} \quad B \supset A$$

Two sets are *equal* if they both contain the same elements; that is,

$$A = B \quad \text{if and only if} \quad A \subset B \text{ and } B \subset A$$

The negations of $p \in A$, $A \subset B$ and $A = B$ are written $p \notin A$, $A \not\subset B$ and $A \neq B$ respectively.

We specify a particular set by either listing its elements or by stating properties which characterize the elements in the set. For example,

$$\boxed{A} = \{1, 3, 5, 7, 9\}$$

means A is the set consisting of the numbers 1, 3, 5, 7 and 9; and

$$\boxed{B} = \{x : x \text{ is a prime number}, x < 15\}$$

means that B is the set of prime numbers less than 15. We also use special symbols to denote sets which occur very often in the text. Unless otherwise specified:

$\mathbf{N}$ = the set of positive integers: $1, 2, 3, \ldots$;

$\mathbf{Z}$ = the set of integers: $\ldots, -2, -1, 0, 1, 2, \ldots$;

$\mathbf{Q}$ = the set of rational numbers;

$\mathbf{R}$ = the set of real numbers;

$\mathbf{C}$ = the set of complex numbers.

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$$\mathbf{C} = \text{the set of complex numbers.}$$

We also use $\emptyset$ to denote the *empty* or *null* set, i.e. the set which contains no elements; this set is assumed to be a subset of every other set.

Frequently the members of a set are sets themselves. For example, each line in a set of lines is a set of points. To help clarify these situations, we use the words *class*, *collection* and *family* synonymously with set. The words *subclass*, *subcollection* and *subfamily* have meanings analogous to subset.

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Example A.1: The sets A and B above can also be written as

$$A = \{x \in \mathbf{N} : x \text{ is odd}, x < 10\} \quad \text{and} \quad B = \{2, 3, 5, 7, 11, 13\}$$

Observe that $9 \in A$ but $9 \notin B$, and $11 \in B$ but $11 \notin A$; whereas $3 \in A$ and $3 \in B$, and $6 \notin A$ and $6 \notin B$.

Example A.2: The sets of numbers are related as follows: $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$.

Example A.3: Let $C = \{x : x^2 = 4, x \text{ is odd}\}$. Then $C = \emptyset$, that is, C is the empty set.

Example A.4: The members of the class $\{\{2,3\}, \{2\}, \{5,6\}\}$ are the sets $\{2,3\}$, $\{2\}$ and $\{5,6\}$.

The following theorem applies.

Theorem A.1: Let A, B and C be any sets. Then: (i) $A \subset A$; (ii) if $A \subset B$ and $B \subset A$, then $A = B$; and (iii) if $A \subset B$ and $B \subset C$, then $A \subset C$.

We emphasize that $A \subset B$ does not exclude the possibility that $A = B$. However, if $A \subset B$ but $A \neq B$, then we say that A is a *proper subset* of B. (Some authors use the symbol $\subseteq$ for a subset and the symbol $\subset$ only for a proper subset.)

When we speak of an *indexed set* $\{a_i : i \in I\}$, or simply $\{a_i\}$, we mean that there is a mapping ϕ from the set I to a set A and that the image $\phi(i)$ of $i \in I$ is denoted a_i. The set I is called the *indexing set* and the elements a_i (the range of ϕ) are said to be indexed by I. A set $\{a_1, a_2, \ldots\}$ indexed by the positive integers **N** is called a *sequence*. An indexed class of sets $\{A_i : i \in I\}$, or simply $\{A_i\}$, has an analogous meaning except that now the map ϕ assigns to each $i \in I$ a set A_i rather than an element a_i.

SET OPERATIONS

Let A and B be arbitrary sets. The *union* of A and B, written $A \cup B$, is the set of elements belonging to A or to B; and the *intersection* of A and B, written $A \cap B$, is the set of elements belonging to both A and B:

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \quad \text{and} \quad A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If $A \cap B = \emptyset$, that is, if A and B do not have any elements in common, then A and B are said to be *disjoint*.

We assume that all our sets are subsets of a fixed *universal set* (denoted here by U). Then the *complement* of A, written A^c, is the set of elements which do not belong to A:

$$A^c = \{x \in U : x \notin A\}$$

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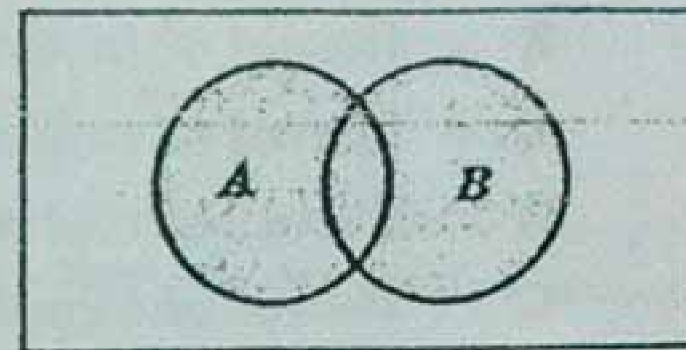
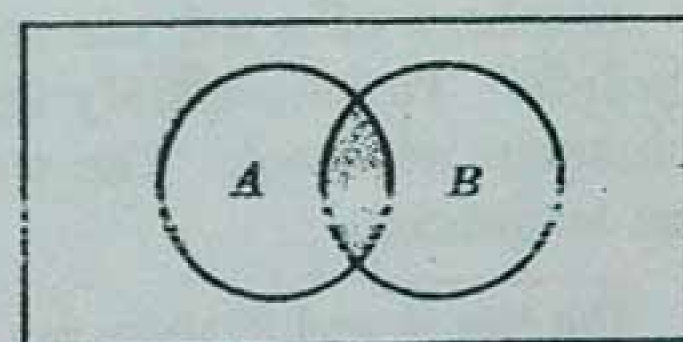
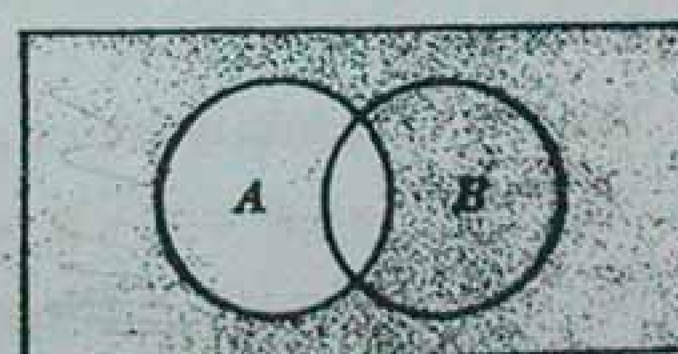


$A \cup B$ is shaded



$A \cap B$ is shaded



A^c is shaded

A'

Sets under the above operations satisfy various laws or identities which are listed in the table below. In fact, we state

Theorem A.2: Sets satisfy the laws in Table 1.

LAWS OF THE ALGEBRA OF SETS	
Idempotent Laws قوانين اللانمو	
1a. $A \cup A = A$	1b. $A \cap A = A$
Associative Laws التجميعية	
2a. $(A \cup B) \cup C = A \cup (B \cup C)$	2b. $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws التبديلية	
3a. $A \cup B = B \cup A$	3b. $A \cap B = B \cap A$
Distributive Laws التوزيعية	
4a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	4b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Laws	
5a. $A \cup \emptyset = A$	5b. $A \cap U = A$
6a. $A \cup U = U$	6b. $A \cap \emptyset = \emptyset$
Complement Laws الإتمام	
7a. $A \cup A^c = U$	7b. $A \cap A^c = \emptyset$
8a. $(A^c)^c = A$	8b. $U^c = \emptyset, \ \emptyset^c = U$
De Morgan's Laws قوانين دومورغان	
9a. $(A \cup B)^c = A^c \cap B^c$	9b. $(A \cap B)^c = A^c \cup B^c$

Table 1

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Remark: Each of the above laws follows from an analogous logical law. For example,

$$A \cap B \;=\; \{x : x \in A \text{ and } x \in B\} \;=\; \{x : x \in B \text{ and } x \in A\} \;=\; B \cap A$$

(Here we use the fact that the composite statement "p and q", written $p \wedge q$, is logically equivalent to the composite statement "q and p", i.e. $q \wedge p$.)

The relationship between set inclusion and the above set operations follows.

Theorem A.3: Each of the following conditions is equivalent to $\boxed{A \subset B}$:

(i) $A \cap B = A$ (iii) $B^c \subset A^c$ (v) $B \cup A^c = U$

(ii) $A \cup B = B$ (iv) $A \cap B^c = \emptyset$

We generalize the above set operations as follows. Let $\{A_i : i \in I\}$ be any family of sets. Then the *union* of the A_i, written $\cup_{i \in I} A_i$ (or simply $\cup_i A_i$), is the set of elements each belonging to at least one of the A_i; and the *intersection* of the A_i, written $\cap_{i \in I} A_i$ or simply $\cap_i A_i$, is the set of elements each belonging to every A_i.

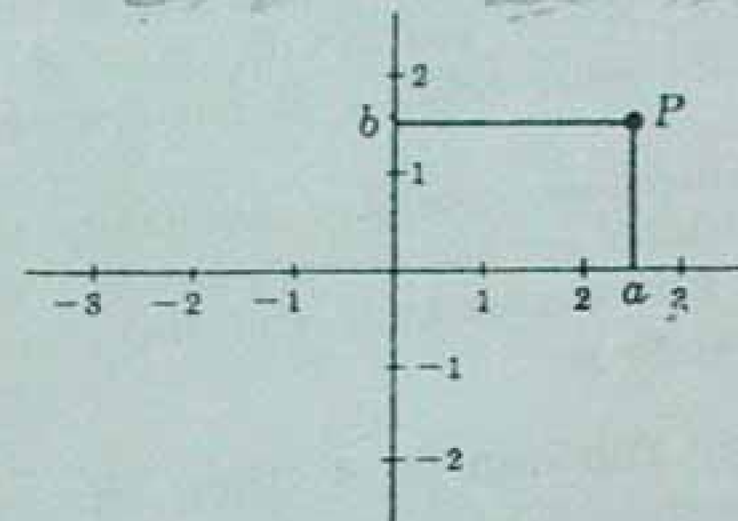
PRODUCT SETS

Let A and B be two sets. The *product set* of A and B, denoted by $A \times B$, consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B \;=\; \{(a, b) : a \in A,\ b \in B\}$$

The product of a set with itself, say $A \times A$, is denoted by A^2.

Example A.6: The reader is familiar with the cartesian plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ as shown below. Here each point P represents an ordered pair (a, b) of real numbers, and vice versa.



Example A.7: Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

Remark: The ordered pair (a, b) is defined rigorously by $(a, b) \equiv \{\{a\}, \{a, b\}\}$. From this definition, the "order" property may be proven; that is, $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

The concept of product set is extended to any finite number of sets in a natural way. The *product set* of the sets $A_1, \ldots, A_m$, written $A_1 \times A_2 \times \cdots \times A_m$, is the set consisting of all m-tuples $(a_1, a_2, \ldots, a_m)$ where $a_i \in A_i$ for each i.

RELATIONS

A *binary relation* or simply *relation* R from a set A to a set B assigns to each ordered pair $(a, b) \in A \times B$ exactly one of the following statements:

(i) "a is related to b", written $a\,R\,b$,

(ii) "a is not related to b", written $a \not R\, b$.

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A relation from a set A to the same set A is called a *relation in A*.

Example A.8: Set inclusion is a relation in any class of sets. For, given any pair of sets A and B, either $A \subset B$ or $A \not\subset B$.

Observe that any relation R from A to B uniquely defines a subset $\hat{R}$ of $A \times B$ as follows:

$$\hat{R} = \{(a, b) : a\,R\,b\}$$

Conversely, any subset $\hat{R}$ of $A \times B$ defines a relation from A to B as follows:

$$a\,R\,b \text{ if and only if } (a, b) \in \hat{R}$$

In view of the above correspondence between relations from A to B and subsets of $A \times B$, we redefine a relation as follows:

Definition: A relation R from A to B is a subset of $A \times B$.

EQUIVALENCE RELATIONS

A relation in a set A is called an *equivalence relation* if it satisfies the following axioms:

[E_1] Every $a \in A$ is related to itself.

[E_2] If a is related to b, then b is related to a.

[E_3] If a is related to b and b is related to c, then a is related to c.

In general, a relation is said to be *reflexive* if it satisfies [E_1], *symmetric* if it satisfies [E_2], and *transitive* if it satisfies [E_3]. In other words, a relation is an equivalence relation if it is reflexive, symmetric and transitive.

Example A.9: Consider the relation $\subset$ of set inclusion. By Theorem A.1, $A \subset A$ for every set A; and if $A \subset B$ and $B \subset C$, then $A \subset C$. That is, $\subset$ is both reflexive and transitive. On the other hand, $\subset$ is not symmetric, since $A \subset B$ and $A \neq B$ implies $B \not\subset A$.

Example A.10: In Euclidean geometry, similarity of triangles is an equivalence relation. For if α, β and γ are any triangles, then: (i) α is similar to itself; (ii) if α is similar to β, then β is similar to α; and (iii) if α is similar to β and β is similar to γ, then α is similar to γ.

If R is an equivalence relation in A, then the *equivalence class* of any element $a \in A$, denoted by $[a]$, is the set of elements to which a is related:

$$[a] = \{x : a\,R\,x\}$$

The collection of equivalence classes, denoted by A/R, is called the *quotient* of A by R:

$$A/R = \{[a] : a \in A\}$$

The fundamental property of equivalence relations follows:

Theorem A.4: Let R be an equivalence relation in A. Then the quotient set A/R is a *partition* of A, i.e. each $a \in A$ belongs to a member of A/R, and the members of A/R are pairwise disjoint.

Example A.11: Let R_5 be the relation in $\mathbf{Z}$, the set of integers defined by

$$x \equiv y \pmod 5$$

which reads "x is congruent to y modulo 5" and which means "$x - y$ is divisible by 5". Then R_5 is an equivalence relation in $\mathbf{Z}$. There are exactly five distinct equivalence classes in $\mathbf{Z}/R_5$:

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$$A_0 = \{\ldots, -10, -5, 0, 5, 10\}$$
$$A_1 = \{\ldots, -9, -4, 1, 6, 11\}$$
$$A_2 = \{\ldots, -8, -3, 2, 7, 12\}$$
$$A_3 = \{\ldots, -7, -2, 3, 8, 13\}$$
$$A_4 = \{\ldots, -6, -1, 4, 9, 14\}$$

Now each integer x is uniquely expressible in the form $x = 5q + r$ where $0 \leq r < 5$; observe that $x \in E_r$ where r is the remainder. Note that the equivalence classes are pairwise disjoint and that $\mathbf{Z} = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$.