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Chapter 1

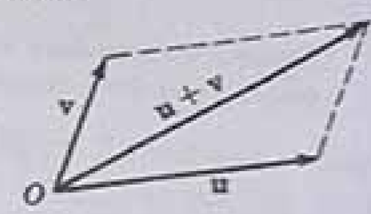
Vectors in R^n and C^n

INTRODUCTION

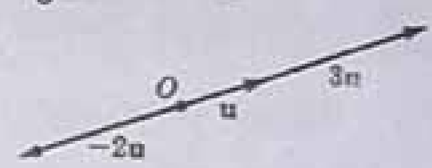
In various physical applications there appear certain quantities, such as temperature and speed, which possess only "magnitude". These can be represented by real numbers and are called *scalars*. On the other hand, there are also quantities, such as force and velocity, which possess both "magnitude" and "direction". These quantities can be represented by arrows (having appropriate lengths and directions and emanating from some given reference point O) and are called *vectors*. In this chapter we study the properties of such vectors in some detail.

We begin by considering the following operations on vectors.

(i) *Addition:* The resultant $\mathbf{u}+\mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ is obtained by the so-called parallelogram law, i.e. $\mathbf{u}+\mathbf{v}$ is the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ as shown on the right.



(ii) *Scalar multiplication:* The product $k\mathbf{u}$ of a real number k by a vector $\mathbf{u}$ is obtained by multiplying the magnitude of $\mathbf{u}$ by k and retaining the same direction if $k \geq 0$ or the opposite direction if $k < 0$, as shown on the right.



Now we assume the reader is familiar with the representation of the points in the plane by ordered pairs of real numbers. If the origin of the axes is chosen at the reference point O above, then every vector is uniquely determined by the coordinates of its endpoint. The relationship between the above operations and endpoints follows.

(i) *Addition:* If (a, b) and (c, d) are the endpoints of the vectors $\mathbf{u}$ and $\mathbf{v}$, then $(a+c, b+d)$ will be the endpoint of $\mathbf{u}+\mathbf{v}$, as shown in Fig. (a) below.

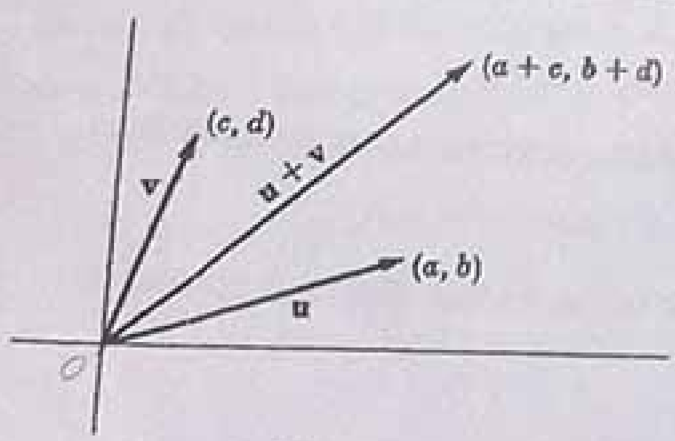


Fig. (a)

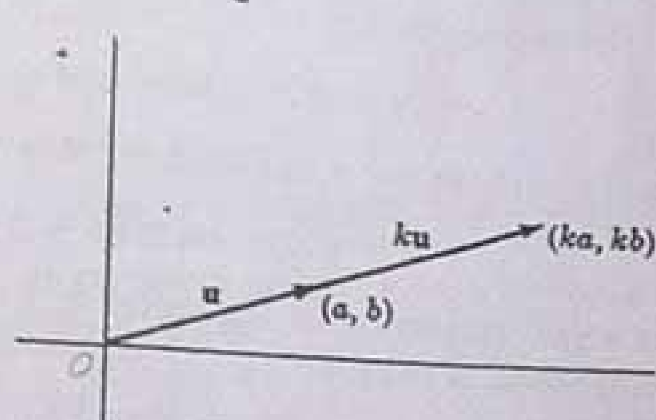


Fig. (b)

(ii) *Scalar multiplication:* If (a, b) is the endpoint of the vector $\mathbf{u}$, then (ka, kb) will be the endpoint of the vector $k\mathbf{u}$, as shown in Fig. (b) above.

Mathematically, we identify a vector with its endpoint; that is, we call the ordered pair (a, b) of real numbers a vector. In fact, we shall generalize this notion and call an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers a vector. We shall again generalize and permit the coordinates of the n-tuple to be complex numbers and not just real numbers. Furthermore, in Chapter 4, we shall abstract properties of these n-tuples and formally define the mathematical system called a *vector space*.

We assume the reader is familiar with the elementary properties of the real number field which we denote by $\mathbf{R}$.

VECTORS IN $\mathbf{R}^n$

The set of all n-tuples of real numbers, denoted by $\mathbf{R}^n$, is called *n-space*. A particular n-tuple in $\mathbf{R}^n$, say

$$u = (u_1, u_2, \ldots, u_n)$$

is called a *point* or *vector*; the real numbers u_i are called the *components* (or: *coordinates*) of the vector u. Moreover, when discussing the space $\mathbf{R}^n$ we use the term *scalar* for the elements of $\mathbf{R}$, i.e. for the real numbers.

Example 1.1: Consider the following vectors:

$$(0, 1), \quad (1, -3), \quad (1, 2, \sqrt{3}, 4), \quad (-5, \tfrac{1}{2}, 0, \pi)$$

The first two vectors have two components and so are points in $\mathbf{R}^2$; the last two vectors have four components and so are points in $\mathbf{R}^4$.

Two vectors u and v are *equal*, written $u = v$, if they have the same number of components, i.e. belong to the same space, and if corresponding components are equal. The vectors $(1, 2, 3)$ and $(2, 3, 1)$ are not equal, since corresponding elements are not equal.

Example 1.2: Suppose $(x - y, x + y, z - 1) = (4, 2, 3)$. Then, by definition of equality of vectors,

$$x - y = 4$$
$$x + y = 2$$
$$z - 1 = 3$$

Solving the above system of equations gives $x = 3$, $y = -1$, and $z = 4$.

VECTOR ADDITION AND SCALAR MULTIPLICATION

Let u and v be vectors in $\mathbf{R}^n$:

$$u = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad v = (v_1, v_2, \ldots, v_n)$$

The *sum* of u and v, written $u + v$, is the vector obtained by adding corresponding components:

$$u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

The *product* of a real number k by the vector u, written ku, is the vector obtained by multiplying each component of u by k:

$$ku = (ku_1, ku_2, \ldots, ku_n)$$

Observe that $u + v$ and ku are also vectors in $\mathbf{R}^n$. We also define

$$-u = -1u \quad \text{and} \quad u - v = u + (-v)$$

The sum of vectors with different numbers of components is not defined.

Example 1.3: Let $u = (1, -3, 2, 4)$ and $v = (3, 5, -1, -2)$. Then

$$u + v = (1+3, -3+5, 2-1, 4-2) = (4, 2, 1, 2)$$
$$5u = (5\cdot 1, 5\cdot(-3), 5\cdot 2, 5\cdot 4) = (5, -15, 10, 20)$$
$$2u - 3v = (2, -6, 4, 8) + (-9, -15, 3, 6) = (-7, -21, 7, 14)$$

Example 1.4: The vector $(0, 0, \ldots, 0)$ in $\mathbf{R}^n$, denoted by $\mathbf{0}$, is called the *zero vector*. It is similar to the scalar 0 in that, for any vector $u = (u_1, u_2, \ldots, u_n)$,

$$u + 0 = (u_1+0, u_2+0, \ldots, u_n+0) = (u_1, u_2, \ldots, u_n) = u$$

Basic properties of the vectors in $\mathbf{R}^n$ under the operations of vector addition and scalar multiplication are described in the following theorem.

Theorem 1.1: For any vectors $u, v, w \in \mathbf{R}^n$ and any scalars $k, k' \in \mathbf{R}$:

(i) $(u+v)+w = u+(v+w)$

(ii) $u+0 = u$

(iii) $u+(-u) = 0$

(iv) $u+v = v+u$

(v) $k(u+v) = ku+kv$

(vi) $(k+k')u = ku+k'u$

(vii) $(kk')u = k(k'u)$

(viii) $1u = u$

Remark: Suppose u and v are vectors in $\mathbf{R}^n$ for which $u = kv$ for some nonzero scalar $k \in \mathbf{R}$. Then u is said to be in the *same direction* as v if $k > 0$, and in the *opposite direction* if $k < 0$.

DOT PRODUCT

Let u and v be vectors in $\mathbf{R}^n$:

$$u = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad v = (v_1, v_2, \ldots, v_n)$$

The *dot* or *inner* product of u and v, denoted by $u\cdot v$, is the scalar obtained by multiplying corresponding components and adding the resulting products:

$$u\cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

The vectors u and v are said to be *orthogonal* (or: *perpendicular*) if their dot product is zero: $u\cdot v = 0$.

Example 1.5: Let $u = (1, -2, 3, -4)$, $v = (6, 7, 1, -2)$ and $w = (5, -4, 5, 7)$. Then

$$u\cdot v = 1\cdot 6 + (-2)\cdot 7 + 3\cdot 1 + (-4)\cdot(-2) = 6 - 14 + 3 + 8 = 3$$
$$u\cdot w = 1\cdot 5 + (-2)\cdot(-4) + 3\cdot 5 + (-4)\cdot 7 = 5 + 8 + 15 - 28 = 0$$

Thus u and w are orthogonal.

Basic properties of the dot product in $\mathbf{R}^n$ follow.

Theorem 1.2: For any vectors $u, v, w \in \mathbf{R}^n$ and any scalar $k \in \mathbf{R}$:

(i) $(u+v)\cdot w = u\cdot w + v\cdot w$

(ii) $(ku)\cdot v = k(u\cdot v)$

(iii) $u\cdot v = v\cdot u$

(iv) $u\cdot u \geq 0$, and $u\cdot u = 0$ iff $u = 0$

Remark: The space $\mathbf{R}^n$ with the above operations of vector addition, scalar multiplication and dot product is usually called *Euclidean n-space*.

NORM AND DISTANCE IN $\mathbf{R}^n$

Let u and v be vectors in $\mathbf{R}^n$: $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. The *distance* between the points u and v, written $d(u, v)$, is defined by

$$d(u, v) = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \cdots + (u_n-v_n)^2}$$

the norm of x

The *norm* (or: *length*) of the vector u, written $\|u\|$, is defined to be the nonnegative square root of $u \cdot u$:

$$\|u\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

By Theorem 1.2, $u \cdot u \geq 0$ and so the square root exists. Observe that

$$d(u, v) = \|u - v\|$$

Example 1.6: Let $u = (1, -2, 4, 1)$ and $v = (3, 1, -5, 0)$. Then

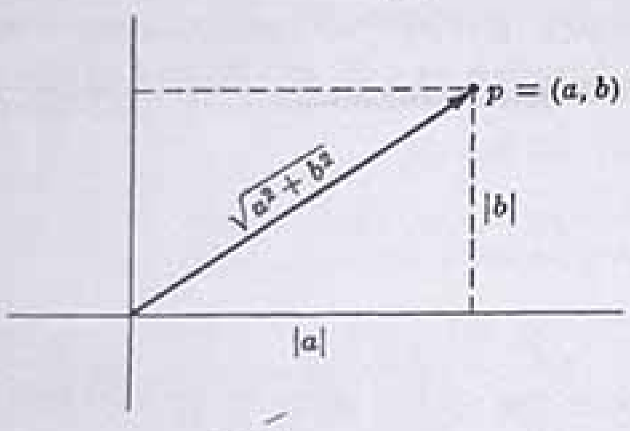
$$d(u, v) = \sqrt{(1-3)^2 + (-2-1)^2 + (4+5)^2 + (1-0)^2} = \sqrt{95}$$

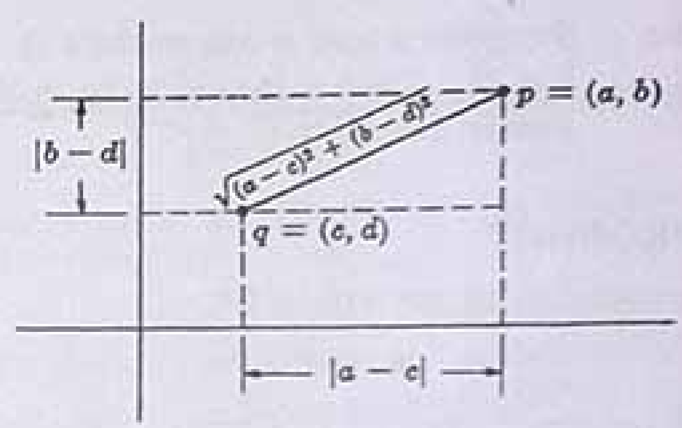
$$\|v\| = \sqrt{3^2 + 1^2 + (-5)^2 + 0^2} = \sqrt{35}$$

Now if we consider two points, say $p = (a, b)$ and $q = (c, d)$ in the plane $\mathbf{R}^2$, then

$$\|p\| = \sqrt{a^2 + b^2} \quad \text{and} \quad d(p, q) = \sqrt{(a-c)^2 + (b-d)^2}$$

That is, $\|p\|$ corresponds to the usual Euclidean length of the arrow from the origin to the point p, and $d(p, q)$ corresponds to the usual Euclidean distance between the points p and q, as shown below:





A similar result holds for points on the line $\mathbf{R}$ and in space $\mathbf{R}^3$. Cubed

Remark: A vector e is called a *unit vector* if its norm is 1: $\|e\| = 1$. Observe that, for any nonzero vector $u \in \mathbf{R}^n$, the vector $e_u = u/\|u\|$ is a unit vector in the same direction as u.

E.Q

over

We now state a fundamental relationship known as the Cauchy-Schwarz inequality.

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Theorem 1.3 (Cauchy-Schwarz): For any vectors $u, v \in \mathbf{R}^n$, $|u \cdot v| \leq \|u\| \, \|v\|$. E.Q

E.Q

Using the above inequality, we can now define the angle θ between any two nonzero vectors $u, v \in \mathbf{R}^n$ by

$$\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}$$

E.Q

Note that if $u \cdot v = 0$, then $\theta = 90°$ (or: $\theta = \pi/2$). This then agrees with our previous definition of orthogonality.

COMPLEX NUMBERS

The set of complex numbers is denoted by $\mathbf{C}$. Formally, a complex number is an ordered pair (a, b) of real numbers; equality, addition and multiplication of complex numbers are defined as follows:

$$(a, b) = (c, d) \quad \text{iff} \quad a = c \text{ and } b = d$$

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

We identify the real number a with the complex number $(a, 0)$:

$$a \leftrightarrow (a, 0)$$

This is possible since the operations of addition and multiplication of real numbers are preserved under the correspondence:

$$(a, 0) + (b, 0) = (a + b, 0) \quad \text{and} \quad (a, 0)(b, 0) = (ab, 0)$$

بالتالي تستبدل مريح متاح / ممسوح

Thus we view **R** as a subset of **C** and replace $(a, 0)$ by a whenever convenient and possible.

يرمز له خاصية

The complex number $(0, 1)$, denoted by i, has the important property that

$$i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1 \quad \text{or} \quad i = \sqrt{-1}$$

إضافة لذلك

Furthermore, using the facts

$$(a, b) = (a, 0) + (0, b) \quad \text{and} \quad (0, b) = (b, 0)(0, 1)$$

we have عبارة/ملاحظة

$$(a, b) = (a, 0) + (b, 0)(0, 1) = a + bi$$

The notation $a + bi$ is more convenient than (a, b). For example, the sum and product of complex numbers can be obtained by simply using the commutative and distributive laws قوانين and $i^2 = -1$:

نحصل عليه التبديل التوزيع

$$(a + bi) + (c + di) = a + c + bi + di = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i$$

مرافق يرمز له يعرف

The *conjugate* of the complex number $z = (a, b) = a + bi$ is denoted and defined by

$$\bar{z} = a - bi$$

إضافة لذلك مقلوب بالتقسيم

(Notice that $z\bar{z} = a^2 + b^2$.) If, in addition, $z \neq 0$, then the inverse z^{-1} of z and division by z are given by

$$z^{-1} = \frac{\bar{z}}{z\bar{z}} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i \quad \text{and} \quad \frac{w}{z} = wz^{-1}$$

where $w \in \mathbf{C}$. We also define

$$-z = -1z \quad \text{and} \quad w - z = w + (-z)$$

بافتراض

Example 1.7: Suppose $z = 2 + 3i$ and $w = 5 - 2i$. Then

$$z + w = (2 + 3i) + (5 - 2i) = 2 + 5 + 3i - 2i = 7 + i$$

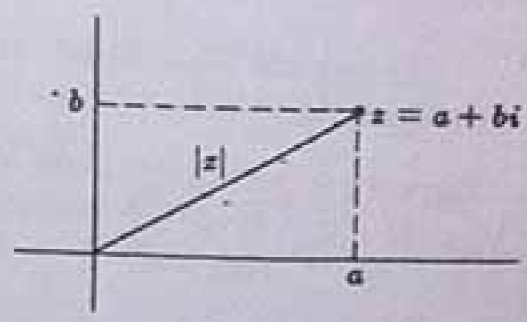
$$zw = (2 + 3i)(5 - 2i) = 10 + 15i - 4i - 6i^2 = 16 + 11i$$

$$\bar{z} = \overline{2 + 3i} = 2 - 3i \quad \text{and} \quad \bar{w} = \overline{5 - 2i} = 5 + 2i$$

$$\frac{w}{z} = \frac{5 - 2i}{2 + 3i} = \frac{(5 - 2i)(2 - 3i)}{(2 + 3i)(2 - 3i)} = \frac{4 - 19i}{13} = \frac{4}{13} - \frac{19}{13}i$$

تمثيلها

Just as the real numbers can be represented by the points on a line, the complex numbers can be represented by the points in the plane. Specifically, we let the point (a, b) in the plane represent the complex number $z = a + bi$, i.e. whose *real part* is a and whose *imaginary part* is b. The *absolute value* القيمة المطلقة of z, written $|z|$, is defined as the distance from z to the origin:



$$|z| = \sqrt{a^2 + b^2}$$

Note that $|z|$ is equal to the norm of the vector (a, b). Also, $|z| = \sqrt{z\bar{z}}$.

Example 1.8: Suppose $z = 2 + 3i$ and $w = 12 - 5i$. Then

$$|z| = \sqrt{4 + 9} = \sqrt{13} \quad \text{and} \quad |w| = \sqrt{144 + 25} = 13$$

ركزنا حقل

ملاحظة ملحق

Remark: In Appendix B we define the algebraic structure called a *field*. We emphasize that the set **C** of complex numbers with the above operations of addition and multiplication is a field.

VECTORS IN C^n

The set of all n-tuples of complex numbers, denoted by C^n, is called *complex n-space*. Just as in the real case, the elements of C^n are called *points* or *vectors*, the elements of **C** are called *scalars*, and *vector addition* in C^n and *scalar multiplication* on C^n are given by

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$$(z_1, z_2, \ldots, z_n) + (w_1, w_2, \ldots, w_n) = (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n)$$

$$z(z_1, z_2, \ldots, z_n) = (zz_1, zz_2, \ldots, zz_n)$$

where $z_i, w_i, z \in \mathbf{C}$.

Example 1.9: $(2+3i, 4-i, 3) + (3-2i, 5i, 4-6i) = (5+i, 4+4i, 7-6i)$

$2i(2+3i, 4-i, 3) = (-6+4i, 2+8i, 6i)$

Now let u and v be arbitrary vectors in C^n:

$$u = (z_1, z_2, \ldots, z_n), \qquad v = (w_1, w_2, \ldots, w_n), \qquad z_i, w_i \in \mathbf{C}$$

نقطي داخلي

E.Q The *dot*, or *inner*, product of u and v is defined as follows:

$$u \cdot v = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$$

Note that this definition reduces to the previous one in the real case, since $w_i = \bar{w}_i$ when w_i is real. The norm of u is defined by

$$\|u\| = \sqrt{u \cdot u} = \sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n} = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$$

Observe that $u \cdot u$ and so $\|u\|$ are real and positive when $u \neq 0$, and 0 when $u = 0$.

Example 1.10: Let $u = (2+3i, 4-i, 2i)$ and $v = (3-2i, 5, 4-6i)$. Then

$$\begin{aligned} u \cdot v &= (2+3i)\overline{(3-2i)} + (4-i)\overline{(5)} + (2i)\overline{(4-6i)} \\ &= (2+3i)(3+2i) + (4-i)(5) + (2i)(4+6i) \\ &= 13i + 20 - 5i - 12 + 8i = 8 + 16i \end{aligned}$$

$$\begin{aligned} u \cdot u &= (2+3i)\overline{(2+3i)} + (4-i)\overline{(4-i)} + (2i)\overline{(2i)} \\ &= (2+3i)(2-3i) + (4-i)(4+i) + (2i)(-2i) \\ &= 13 + 17 + 4 = 34 \end{aligned}$$

$$\|u\| = \sqrt{u \cdot u} = \sqrt{34}$$

E.Q The space C^n with the above operations of vector addition, scalar multiplication and dot product, is called *complex Euclidean n-space*.

Remark: If $u \cdot v$ were defined by $u \cdot v = z_1w_1 + \cdots + z_nw_n$, then it is possible for $u \cdot u = 0$ even though $u \neq 0$, e.g. if $u = (1, i, 0)$. In fact, $u \cdot u$ may not even be real.

ملاحظة على الرغم