٧/٤/٢٠١٤

المحاضرة الثامنة

التوابع القطعية

$$\operatorname{ch} x = \frac{e^{x} + e^{-x}}{2}$$

$$\operatorname{sh} x = \frac{e^{x} - e^{-x}}{2}$$

$$\operatorname{Th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} , \quad \operatorname{Coth} x = \frac{\operatorname{ch} x}{\operatorname{sh} x}$$

الاشتقاقات:

$$(\operatorname{ch} x)' = \operatorname{sh} x$$

$$(\operatorname{sh} x)' = \operatorname{ch} x$$

$$(\operatorname{Th} x)' = \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{ch}^2 x} = \frac{1}{\operatorname{ch}^2 x}$$

$$= 1 - \operatorname{th}^2 x$$

$$(\operatorname{coth} x)' = \frac{\operatorname{sh}^2 x - \operatorname{ch}^2 x}{\operatorname{sh}^2 x} = -\frac{1}{\operatorname{sh}^2 x}$$

$$= 1 - \operatorname{coth}^2 x$$

علاقات شهيرة:

$$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$$

$$\operatorname{ch}^2 x + \operatorname{sh}^2 x = \operatorname{ch} 2x$$

$$2 \operatorname{sh} x . \operatorname{ch} x = \operatorname{sh} 2x$$

$$\operatorname{ch}(x \mp y) = \operatorname{ch} x . \operatorname{ch} y \mp \operatorname{sh} x \operatorname{sh} y .$$

$$\operatorname{sh}(x \mp y) = \operatorname{sh} x \operatorname{ch} y \mp \operatorname{ch} x . \operatorname{sh} y$$

$$y = \operatorname{arc\,sh} x \Rightarrow x = \operatorname{sh} y$$

$$1 = \operatorname{ch} y \, y' \Rightarrow y' = \frac{1}{\operatorname{ch} y}$$

$$\operatorname{ch}^2 y - \operatorname{sh}^2 y = 1$$

$$\operatorname{ch}^2 y = 1 + \operatorname{sh}^2 y$$

$$\operatorname{ch} y = \sqrt{1 + x^2} \Rightarrow \boxed{y' = \frac{1}{\sqrt{1+x^2}}}$$

$$y = \operatorname{arc\,ch} x \Rightarrow x = \operatorname{ch} y$$

$$1 = \operatorname{sh} y \, y' \Rightarrow$$

$$y' = \frac{1}{\operatorname{sh} y}$$

$$\operatorname{ch}^2 y - \operatorname{sh}^2 y = 1 \Rightarrow$$

$$\operatorname{ch}^2 y - 1 = \operatorname{sh}^2 y \Rightarrow \operatorname{sh} y = \sqrt{x^2 - 1}$$

$$\Rightarrow \boxed{y' = \frac{1}{\sqrt{x^2 - 1}}}$$

$$y = \operatorname{arc\,sh} x \Rightarrow y' = \frac{1}{\sqrt{x^2 + 1}}$$

$$y = \operatorname{arc\,ch} x \Rightarrow y' = \frac{1}{\sqrt{x^2 - 1}}$$

$$y = \operatorname{arc\,th} x \Rightarrow x = \operatorname{th} y$$

$$1 = (1 - \operatorname{Th}^2 y) y'$$

$$y' = \frac{1}{1 - \operatorname{th}^2 y} = \boxed{\frac{1}{1 - x^2}}$$

$$y = \operatorname{arc\,coth} x \Rightarrow x = \operatorname{coth} y$$

حل التكاملات التالية:

$I_1=\int \frac{1}{ch x}\,dx$ ، $I_2=\int \frac{1}{sh x}\,dx$

$I_3=\int \frac{sh x+1}{ch x}\,dx$ ، $I_4=\int \frac{dx}{(1-ch x)^2}$

$I_5=\int ch^2 x\, sh^3 x\,dx$

$I_6=\int \frac{ch^3 x}{1-th x}\,dx$

$I_7=\int \frac{1}{ch 2x+sh 2x}\,dx$

$I_8=\int \frac{1}{\cos 2x+\sin 2x}\,dx.$

الحل: لحل التكاملات السابقة نقوم بفرض:

$t=th\frac{x}{2}$

وهذه الفرضية من اجل

I_1, I_2, I_3, I_4

ولأجل I_7 نفرض $T=Th x$

ولأجل I_8 نفرض $T=\tan x.$

$\int (sh x\, ch x)\,dx$

$t=th\frac{x}{2} \Rightarrow \frac{x}{2}=\text{arc}\,th\, t$

$\Rightarrow x=2\,\text{arc}\,th\, t$

$dt=\frac{1}{2}\left(1-th^2\frac{x}{2}\right)dx$

$$\Rightarrow \boxed{dx=\frac{2dt}{1-t^2}}$$

$y=\text{arc}\,coth x \Rightarrow$

$x=coth y$

$1=(1-coth^2 y)\,y'$

$$\Rightarrow y'=\frac{1}{1-coth^2 y} \Rightarrow \boxed{y'=\frac{1}{1-x^2}}$$

التكاملات:

$I_1=\int sh x\,dx=ch x+c$

$I_2=\int ch x\,dx=sh x+c$

$I_3=\int Th x\,dx=\ln|ch x|+c$

$I_4=\int coth x\,dx=\ln|sh x|+c$

$I_5=\int \frac{1}{ch^2 x}\,dx=Th x+c$

$I_6=\int \frac{1}{sh^2 x}\,dx=-coth x+c$

$I_7=\int \frac{1}{\sqrt{x^2\mp 1}}\,dx=\ln\left|x+\sqrt{x^2\mp 1}\right|+c$

لو كانت بدل $x^2+1 \leftarrow (x^2+a^2)$

لأصبح التكامل من الشكل:

$=\ln\left|\frac{x}{a}+\frac{1}{a}\sqrt{x^2\mp a^2}\right|+c$

$I_8=\int \frac{dx}{\sqrt{x^2+1}}=\text{arc}\,sh x+c$

$I_9=\int \frac{dx}{\sqrt{x^2-1}}=\text{arc}\,ch x+c$

$I_{10}=\int \frac{dx}{1-x^2}=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+c$

تابع

$$4\text{-})\int \frac{1}{\left(\frac{-2t^2}{1-t^2}\right)^2}\cdot\frac{2dt}{1-t^2}$$

$$=2\int \frac{1-t^2}{4t^4}dt$$

$$=\frac{1}{2}\left[\int t^{-4}dt-\int t^{-2}dt\right]$$

$$=-\frac{1}{6}\frac{1}{t^3}+\frac{1}{2}\frac{1}{t}+c$$

$$t=\text{th}\frac{x}{2}$$

$$I_5=\int ch^2x\, sh^3x\, dx.$$

$$=\int ch^2x\, sh^2x\, shx\, dx$$

$$=\int ch^2x(ch^2x-1)shx\, dx$$

$$=\int (ch^4x-ch^2x)shx\, dx$$

$$=\frac{1}{5}ch^5x-\frac{1}{3}ch^3x+c$$

I_6 نتركه مسيرة

$$\int \frac{(1+\text{th}x)ch^3x}{1-\text{th}^2x}dx$$

$(1+\text{th}x)$ ← → $(1-\text{th}x)$

$$=\int \frac{ch^3x+shx\,ch^2x}{\frac{1}{ch^2x}}dx$$

$$chx=\frac{ch^2\frac{x}{2}+sh^2\frac{x}{2}}{\left(ch^2\frac{x}{2}-sh^2\frac{x}{2}\right)}$$

$$\Rightarrow \left\{chx=\frac{1+t^2}{1-t^2}\right\}$$

$$shx=\frac{2\,sh\frac{x}{2}\cdot ch\frac{x}{2}}{\left(ch^2\frac{x}{2}-sh^2\frac{x}{2}\right)}$$

$ch^2\frac{x}{2}$ نقسم البسط والمقام

$$\Rightarrow \left\{shx=\frac{2t}{1-t^2}\right\}$$

$$I_1=\int \frac{1}{\frac{1+t^2}{1-t^2}}\cdot\frac{2dt}{1-t^2}=\int\frac{2dt}{1+t^2}$$

$$=2\arctan t+c$$

$$\left\{I_1=2\arctan\left(\text{th}\frac{x}{2}\right)+c\right\}$$

$$I_2=\int \frac{1}{\frac{2t}{1-t^2}}\cdot\frac{2dt}{1-t^2}=\ln|t|+c$$

$$I_2=\ln\left|\text{th}\frac{x}{2}\right|+c$$

$$I_3=\int \text{Th}x\, dx+\int\frac{1}{chx}dx$$

$$I_3=\ln|chx|+2\arctan\left(\text{th}\frac{x}{2}\right)+c$$

$$I_4=\int \frac{1}{\left[\frac{1-t^2}{1-t^2}-\frac{1+t^2}{1-t^2}\right]^2}\cdot\frac{2dt}{1-t^2}$$

$$\frac{1}{ch^2x} = 1 - Th^2x$$

$$Thx = \frac{shx}{chx}$$

نقوم بالتعويض

$$= \int (ch^5x + \underbrace{shx\, ch^4x}_{\frac{1}{5}ch^5x})\, dx$$

$$I = \int ch^5x\, dx = \int ch^4x\, chx\, dx$$

$$= \int (ch^2x)^2\, chx\, dx$$

$$= \int (1 + sh^2x)^2\, chx\, dx$$

$$I = \int (1 + 2sh^2x + sh^4x)\, chx\, dx$$

$$= \int chx\, dx + \int 2sh^2x\, chx\, dx$$

$$+ \int sh^4x\, chx\, dx$$

$$= shx + \frac{2}{3}sh^3x + \frac{1}{5}sh^5x + C$$

$$I_7 = \int \frac{1}{ch2x + sh^2x}\, dx$$

نوضن $T = Thx$

$$dt = (1 - Th^2x)\, dx \Rightarrow$$

$$(1 - t^2)\, dx \Rightarrow \boxed{dx = \frac{dt}{1-t^2}}$$

$$ch2x = ch^2x + sh^2x$$

نشع هنا لت نوضن ch^2x

$$1 - t^2 = \frac{1}{ch^2x}$$

$$ch^2x = \frac{1}{1-t^2}$$ نقلب العلاقة

ولدينا أيضاً

$$sh^2x = ch^2x - ch^2x$$

$$sh^2x = \frac{1}{1-t^2} - \frac{1-t^2}{1-t^2}$$

وبالتعويض

$$ch2x = \frac{1}{1-t^2} + \frac{t^2}{1-t^2}$$

$$\Rightarrow \boxed{ch2x = \frac{1+t^2}{1-t^2}}$$

$$sh2x = 2shx\, chx$$

$$\Rightarrow \boxed{sh2x = 2\frac{t}{1-t^2}}$$

نعوض في I_7 ونكمل

انتهت المحاضرة