X

4/5/2014 — المحاضرة الثانية عشر

جداء الفئات

تعريف: لتكن $(\mathcal{L}_i)_{i\in I}$ أسرة من الفئات، نرمز لجداء هذه الفئات بالشكل $\prod_{i\in I}\mathcal{L}_i$;

$$\prod_{i\in I}\mathcal{L}_i \xrightarrow{I=\{1,2\}} \mathcal{L}_1\times\mathcal{L}_2$$

وتسمى فئة الجداء وتتألف من:

1- صف الأشياء: أشياء هذه الفئة جميع الأسر المرقمة من الشكل

$$(X_i)_{i\in I} \; : \; \forall i\in I \; ; \; X_i\in ob(\mathcal{L}_i)$$

I مجموعة من الأدلة.

2- صف المورفيزمات: نعرض $(X_i)_{i\in I}$, $(Y_i)_{i\in I}$

المورفيزم: $f:(X_i)\longrightarrow(Y_i)$ هو أسرة المورفيزمات.

$$f=(f_i)_{i\in I} \; : \; \forall i\in I \; ; \; f_i\in\mathcal{L}_i(X_i,Y_i)$$

* باعتراض $I=\{1,2\}$ التأكد من شروط الفئة..

مبرهنة: لتكن $\mathcal{L}_1,\mathcal{L}_2$ فئتين، $F:\mathcal{L}_1\longrightarrow\mathcal{L}_2$

$G:\mathcal{L}_2\longrightarrow\mathcal{L}_1$

دوال مباشرة عندئذ يوجد:

① دالي مباشر، $Hom_{\mathcal{L}_1}(G):\mathcal{L}_1^\circ\times\mathcal{L}_1\longrightarrow Sets$

② دالي مباشر [illegible]، $Hom_{\mathcal{L}_2}(F):\mathcal{L}_1^\circ\times\mathcal{L}_2\longrightarrow Sets$

الإثبات:

① لنعرف $Hom_{\mathcal{L}_1}(G):\mathcal{L}_1^\circ\times\mathcal{L}_2\longrightarrow Sets$

$\forall (A,B)\in ob(\mathcal{L}_1^\circ,\mathcal{L}_2)$

حيث

$A\in ob(\mathcal{L}_1^\circ)=ob(\mathcal{L}_1)$

$B\in ob(\mathcal{L}_2)\Rightarrow G(B)\in ob(\mathcal{L}_1)$

$$Hom(G)(A,B) = \mathcal{L}_1(\underbrace{A, G(B)}_{\in Sets})$$

← دالي يؤثر على المتغير الثاني

نثبت أن $Hom_{\mathcal{L}_1}(G)$ هو تطبيق أشياء

$(A_1,B_1) = (A_2,B_2)$

$A_1 = A_2$

$B_1 = B_2$

$\Rightarrow G(B_1) = G(B_2)$

ليكن: $(A_1,A_2),(B_1,B_2) \in ob(\mathcal{L}_1^\circ \times \mathcal{L}_2)$

و: $(f_1,f_2) : (A_1,A_2) \longrightarrow (B_1,B_2)$

نثبت أن: $f_1 : A_1 \longrightarrow B_1 \in \mathcal{L}_1^\circ(A_1,B_1)$

$f_2 : A_2 \longrightarrow B_2 \in \mathcal{L}_2(A_2,B_2)$

لنضع:

$$Hom_{\mathcal{L}_1}(G)(f_1,f_2) : Hom_{\mathcal{L}_1}(G)(A_1,A_2) \longrightarrow Hom_{\mathcal{L}_1}(G)(B_1,B_2)$$

$$: \mathcal{L}_1(A_1, G(A_2)) \longrightarrow \mathcal{L}_1(B_1, G(B_2))$$

لكل مورفيزم $U : A \to B$

دالي مباشر $F(u) : F(A) \to F(B)$

لنضع: $u : A_1 \longrightarrow G(A_2) \in \mathcal{L}_1(A_2, G(A_2))$

$$\boxed{Hom_{\mathcal{L}_1}(G)(f_1,f_2)(u) = G(f_2).u.f_1 \in \mathcal{L}_1(B_1, G(B_2))}$$

$$\begin{tikzcd}
B_1 \arrow[r, "f_1"] \arrow[drr, "G(f_2).u.f_1"'] & A_1 \arrow[d, "u"] \\
 & G(A_2) \arrow[d, "G(f_2)"] \\
 & G(B_2)
\end{tikzcd}$$

$G : \mathcal{L}_2 \to \mathcal{L}_1$

$G(f_2) : G(A_2) \to G(B_2)$

المورفيزمات f_1 ، u ، $G(f_2)$

من نفس الفئة وهي $\mathcal{L}_1$

فالتركيب ممكن.

نثبت أنه تطبيق.

ليكن: $A_1, A_2 \in \mathcal{L}_1^\circ \times \mathcal{L}_2$,

$I_{(A_1,A_2)} : (A_1,A_2) \longrightarrow (A_1,A_2)$

المورفيزم المطابق في $\mathcal{L}_1^\circ \times \mathcal{L}_2$: $I_{(A_1,A_2)} = (I_{A_1}, I_{A_2})$ «حسب تعريف الجداء ((1))

×

$$Hom_{\mathcal{L}_1}(G)(I_{(A_1,A_2)}) : Hom_{\mathcal{L}_1}(G)(A_1,A_2) \longrightarrow Hom(G)(A_1,A_2)$$
$$: \mathcal{L}_1(A_1, G(A_2)) \longrightarrow \mathcal{L}_1(A_1, G(A_2))$$

أولاً كان:
$$w : A_1 \longrightarrow G(A_2) \in \mathcal{L}_1(A, G(A_2))$$

فإن:
$$Hom_{\mathcal{L}_1}(G)(I_{(A_1,A_2)})(w) = G(I_{A_2}) \cdot w \cdot I_{A_1}$$
$$= I_{G(A_2)} \cdot w : w$$

ومنه نجد أن:
$$Hom_{\mathcal{L}_1}(G)(I_{(A_1,A_2)}) = I_{Hom_{\mathcal{L}_1}(G)(A_1,A_2)}$$

* ليكن:
$$f : A \longrightarrow B \qquad g : B \longrightarrow D$$

مورفيزمات للفئة $\mathcal{L}_1^{\circ} \times \mathcal{L}_2$ عندئذ:

$g \cdot f : A \longrightarrow D$ مورفيزم للفئة $\mathcal{L}_1^{\circ} \times \mathcal{L}_2$

$$g \cdot f = (g_1, g_2) \cdot (f_1, f_2) = (g_1 \cdot f_1, g_2 \cdot f_2)$$

$$g_1 \cdot f_1 : A_1 \longrightarrow D_1 \in \mathcal{L}_1^{\circ}(A_1, D_1)$$
$$g_2 \cdot f_2 : A_2 \longrightarrow D_2 \in \mathcal{L}_2(A_2, D_2$$

$$Hom_{\mathcal{L}_1}(G)(g \cdot f) : Hom_{\mathcal{L}_1}(G)(A_1,A_2) \longrightarrow Hom_{\mathcal{L}_1}(G)(D_1, D_2)$$
$$: \mathcal{L}_1(A_1, G(A_2)) \longrightarrow \mathcal{L}_1(D_1, G(D_2))$$

إذا كان:
$$(\sigma : (A_1 \longrightarrow G(A_2)) \longrightarrow$$

ليكن: $\sigma : A_1 \longrightarrow G(A_2) \in \mathcal{L}_1(A_1, G(A_2))$

لذلك: $\mathrm{Hom}_{\mathcal{L}_1}(G)(g.f)(\sigma) = G(g_2.f_2).\sigma.(f_1, g_1)$

$$\mathcal{L}_1 \ni \quad \begin{tikzcd} B_1 \arrow[r, "f_1"] & A_1 \arrow[dl, "f_1.g_1"] \\ D_1 \arrow[u, "g_1"] & \end{tikzcd} \qquad \mathcal{L}_1^o \ni \quad \begin{tikzcd} A_1 \arrow[r, "f_1 \in \mathcal{L}_1^o"] \arrow[dr, "g_1.f_1 \in \mathcal{L}_1^o"'] & B_1 \arrow[d, "g_1 \in \mathcal{L}_1^o"] \\ & D_1 \end{tikzcd}$$

$$\mathrm{Hom}_{\mathcal{L}_1}(G)(g.f)(\sigma) = G(g_2).(G(f_2).\sigma.f_1).g_1$$

$$G(f_2).\sigma.f_1 : B \xrightarrow{f_1} A_1 \xrightarrow{\sigma} G(A_2) \xrightarrow{G(f_2)} G(B_2)$$
$$G(f_2).\sigma.f_1 : B \longrightarrow G(B_2)$$

$$\mathrm{Hom}_{\mathcal{L}_1}(G)(g_1, g_2) : \mathrm{Hom}_{\mathcal{L}_1}(G)(B_1, B_2) \longrightarrow \mathrm{Hom}_{\mathcal{L}_1}(G)(D_1, D_2)$$
$$= \mathcal{L}_1(B_1, G(B_2)) \longrightarrow \mathcal{L}_1(D_1, G(D_2))$$

$$G(f_2).\sigma.f_1 \in \mathcal{L}_1(B_1, G(B_2))$$

$$\mathrm{Hom}_{\mathcal{L}_1}(G)(g)(G(f_2).\sigma.f_1) = G(g_2).(G(f_2).\sigma.f_1).g_1$$

وهكذا نجد أن:

$$\mathrm{Hom}_{\mathcal{L}_1}(G)(g.f)(\sigma) = \mathrm{Hom}_{\mathcal{L}_1}(G)(g)(G(f_2).\sigma.f_1)$$
$$= \mathrm{Hom}_{\mathcal{L}_1}(G)(g)(\mathrm{Hom}_{\mathcal{L}_1}(G)(f)(\sigma))$$

$$\mathrm{Hom}_{\mathcal{L}_1}(G)(g.f)(\sigma) = (\mathrm{Hom}_{\mathcal{L}_1}(G)(g).\mathrm{Hom}_{\mathcal{L}_1}(G)(f))(\sigma)$$

هذا الكلام صحيح $\forall \sigma$

ومنه:

$$\mathrm{Hom}_{\mathcal{L}_1}(G)(g.f) = \mathrm{Hom}_{\mathcal{L}_1}(G)(g).\mathrm{Hom}_{\mathcal{L}_1}(G)(f)$$

$$Hom_{\mathcal{C}_2}(F) : \mathcal{C}_1^{\circ} \times \mathcal{C}_2^{\circ} \longrightarrow Sets$$ ②

دالي مباشر $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$.

ليكن $(A,B) \in ob(\mathcal{C}_1^{\circ} \times \mathcal{C}_2)$

لنعرف: $Hom_{\mathcal{C}_2}(F)(A,B) = \mathcal{C}_2(F(A), B) \in Sets$

وهو تطبيق الاسناد

$$A \in ob(\mathcal{C}_1^{\circ}) = ob(\mathcal{C}_1) \Rightarrow F(A) \in ob(\mathcal{C}_2) \,,\; B \in ob(\mathcal{C}_2)$$

* ليكن: $f : A \longrightarrow B \in Mor(\mathcal{C}_1^{\circ} \times \mathcal{C}_2)$

$$A = (A_1, A_2) \quad , B = (B_1, B_2) \quad , f = (f_1, f_2)$$

$$f_1 : A_1 \longrightarrow B_1 \in \mathcal{C}_1^{\circ}(A_1, B_1)$$
$$f_1 : B_1 \longrightarrow A_1 \in \mathcal{C}_1(B_1, A_1)$$

$$f_2 : A_2 \longrightarrow B_2 \in \mathcal{C}_2(A_2, B_2)$$

$$Hom_{\mathcal{C}_2}(F)(f) : Hom_{\mathcal{C}_2}(F)(A) \longrightarrow Hom_{\mathcal{C}_2}(F)(B)$$
$$: \mathcal{C}_2(F(A_1), A_2) \longrightarrow \mathcal{C}_2(F(B_1), B_2)$$

لأي $\nu : F(A_1) \longrightarrow A_2 \in \mathcal{C}_2(F(A_1), A_2)$

لنضع: $Hom(F)(f)(\nu) = f_2 . \nu . F(f_1)$

$$\begin{tikzcd}
F(B_1) \arrow[r, "F(f_1)"] \arrow[ddr, "f_2 . \nu . F(f_1)"'] & F(A_1) \arrow[d, "\nu"] \\
 & A_2 \arrow[d, "f_2"] \\
 & B_2
\end{tikzcd}$$

لدينا $I_A : A \longrightarrow A$

$$A = (A_1, A_2) \Rightarrow I_A = (I_{A_1}, I_{A_2})$$

$$\begin{aligned} Hom_{\mathcal{L}_2}(F)(I_A) &: Hom_{\mathcal{L}_2}(F)(A_1, A_2) \longrightarrow Hom_{\mathcal{L}_2}(F)(A_1, A_2) \\ &: \mathcal{L}_2(F(A_1), A_2) \longrightarrow \mathcal{L}_2(F(A_1), A_2) \end{aligned}$$

$\forall\, w : F(A_1) \longrightarrow A_2 \in \mathcal{L}_2(F(A_1), A_2)$

$$\begin{aligned} Hom_{\mathcal{L}_2}(F)(I_A)(w) &= I_{A_2} . w . F(I_{A_1}) \\ &= w . I_{F(A_1)} \\ &= w \end{aligned}$$

ومنه $Hom_{\mathcal{L}_2}(F)(I_A) = I_{Hom_{\mathcal{L}_2}(F)(A)}$

* ليكن $f : A \longrightarrow B$, $g : B \longrightarrow D$ مورفيزمات الفئة $\mathcal{L}_1^{\circ} \times \mathcal{L}_2$

$$g.f : A \longrightarrow D \in Mor(\mathcal{L}_1^{\circ} \times \mathcal{L}_2)$$

$A = (A_1, A_2)$, $B = (B_1, B_2)$, $D = (D_1, D_2)$

$f = (f_1, f_2)$, $g = (g_1, g_2)$

$$g.f = (\underbrace{g_1 . f_1}_{\in \mathcal{L}_1^{\circ}} , \underbrace{g_2 . f_2}_{\in \mathcal{L}_2})$$

$$\begin{aligned} Hom_{\mathcal{L}_2}(F)(g.f) &: Hom_{\mathcal{L}_2}(F)(A) \longrightarrow Hom_{\mathcal{L}_2}(F)(D) \\ &: \mathcal{L}_2(F(A), A_2) \longrightarrow \mathcal{L}_2(F(D_1), D_2) \end{aligned}$$

$\forall\, \lambda : F(A_1) \longrightarrow A_2 \in \mathcal{L}_2(F(A_1), A_2)$

$$Hom_{\mathcal{L}_2}(F)(g.f)(\lambda) = (g_2 . f_2) . \lambda . F(f_1 . g_1)$$

$g_1 . f_1 : A_1 \longrightarrow D_1 \in \mathcal{L}_1^{\circ}(A_1, D_1)$

$f_1 . g_1 : D_1 \longrightarrow A_1 \in \mathcal{L}_1(A_1, D_1)$

$$= g_2 . \overbrace{(f_2 . \lambda . F(f_1))}^{\in \mathcal{L}_2(F(B_1), B_2)} . F(g_1)$$

$$\begin{tikzcd} B_1 \arrow[r, "f_1"] & A_1 \\ D_1 \arrow[u, "g_1"] \arrow[ur, "f_1 . g_1 \in \mathfrak{f}_1"'] & \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[dr, "g_1 . f_1 \in \mathfrak{f}_1"'] & B_1 \arrow[d] \\ & D_1 \end{tikzcd}$$

$$f_2 . \lambda . F(f_1) : F(B_1) \xrightarrow{F(f_1)} F(A_1) \xrightarrow{\lambda} A_2 \xrightarrow{f_2} B_2$$

$$f_2 . \lambda . F(f_1) : F(B_1) \longrightarrow B_2$$

$$\mathrm{Hom}_{\mathfrak{f}_1}(F)(g) : \mathrm{Hom}_{\mathfrak{f}_1}(F)(B) \longrightarrow \mathrm{Hom}_{\mathfrak{f}_2}(F)(D)$$

$$: \mathfrak{f}_2(F(B_1), B_2) \longrightarrow \mathfrak{f}_2(F(D_1), D_2)$$

وهكذا نجد ان:

$$\mathrm{Hom}_{\mathfrak{f}_2}(F)(g.f)(\lambda) = \mathrm{Hom}_{\mathfrak{f}_2}(F)(g)(f_2 . \lambda . F(f_1))$$

$$= \mathrm{Hom}_{\mathfrak{f}_2}(F)(g)(\mathrm{Hom}_{\mathfrak{f}_2}(F)(f)(\lambda))$$

$$= \mathrm{Hom}_{\mathfrak{f}_2}(F)(g) . \mathrm{Hom}_{\mathfrak{f}_2}(F)(f)\}(\lambda)$$

$$\mathrm{Hom}_{\mathfrak{f}_2}(F)(g.f) = \mathrm{Hom}_{\mathfrak{f}_2}(F)(g) . \mathrm{Hom}_{\mathfrak{f}_2}(F)(f)$$

انتهت المحاضرة