مثال: حل المعادلة التكاملية مستخدماً طريقتي التقريبات المتتالية و النوى المكررة.

$$\Psi(x) = 1 + \lambda \int_0^1 (1-3xt)\,\Psi(t)\,dt$$

الحل:

أولاً: التقريبات المتتالية: أولاً نتحقق من الشروط:

$h(x)=1$ كمولة تربيعياً وكذلك $k(x,t)=1-3xt$ كمولة تربيعياً.

نحسب B^2:

$$B^2 = \int_0^1\!\!\int_0^1 |1-3xt|^2\,dx\,dt \quad ; \quad |\lambda| < \frac{1}{B}$$

$$B^2 = \frac{1}{2} \quad ; \quad |\lambda| < \frac{1}{B} \Rightarrow |\lambda| < \sqrt{2}$$

$$*\ \Psi_0(x) = h(x) = 1$$

$$*\ \Psi_1(x) = h(x) + \lambda \int_a^b k(x,t)\,\Psi_0(t)\,dt = 1 + \lambda \int_0^1 (1-3xt)(1)\,dt$$

$$\Psi_1(x) = 1 + \lambda\left(1 - \frac{3}{2}x\right)$$

$$*\ \Psi_2(x) = h(x) + \lambda \int_a^b k(x,t)\,\Psi_1(t)\,dt = 1 + \lambda \int_0^1 (1-3xt)\left[1 + \lambda\left(1 - \frac{3}{2}t\right)\right] dt$$

$$\Psi_2(x) = 1 + \lambda \int_0^1 (1-3xt)\left[1 + \lambda - \frac{3\lambda t}{2}\right] dt$$

بفك الأقواس نجد:

$$\Psi_2(x) = 1 + \lambda \int_0^1 \left[1 + \lambda - \frac{3\lambda t}{2} - 3xt - 3x\lambda t + \frac{9x\lambda t^2}{2}\right] dt$$

نكامل:

$$\Psi_2(x) = 1 + \lambda\left[t + \lambda t - \frac{3\lambda t^2}{4} - \frac{3xt^2}{2} - \frac{3x\lambda t^2}{2} + \frac{3x\lambda t^3}{2}\right]_0^1$$

$$\Psi_2(x) = 1 + \lambda\left[1 + \lambda - \frac{3}{4}\lambda - \frac{3}{2}x - \frac{3}{2}x\lambda + \frac{3}{2}x\lambda\right]$$

$$\Psi_2(x) = 1 + \lambda + \lambda^2 - \frac{3}{4}\lambda^2 - \frac{3}{2}x\lambda \Rightarrow \Psi_2(x) = 1 + \lambda + \frac{1}{4}\lambda^2 - \frac{3}{2}x\lambda$$

Subject

$$\Psi_2(x) = 1 + \lambda\left(1 - \frac{3}{2}x\right) + \frac{1}{4}\lambda^2$$

$$* \quad \Psi_3(x) = 1 + \lambda\int_a^b k(x,t)\,.\,\Psi_2(t)\,dt$$

$$\Psi_3(x) = 1 + \lambda\int_0^1 (1-3xt)\left[1 + \lambda\left(1 - \frac{3}{2}t\right) + \frac{1}{4}\lambda^2\right]dt$$

وبحساب التكامل نجد :

$$\Psi_3(x) = 1 + \lambda\left(1 - \frac{3}{2}x\right) + \frac{1}{4}\lambda^2 + \frac{1}{4}\lambda^3\left(1 - \frac{3}{2}x\right)$$

$$* \quad \Psi_4(x) = 1 + \lambda\left(1 - \frac{3}{2}x\right) + \frac{1}{4}\lambda^2 + \frac{1}{4}\lambda^3\left(1 - \frac{3}{2}x\right) + \left(\frac{\lambda^2}{4}\right)^2$$

$$\Psi_4(x) = 1 + \left(1 - \frac{3}{2}x\right)\left(\lambda + \frac{1}{4}\lambda^3\right) + \frac{\lambda^2}{4}\left(1 + \frac{\lambda^2}{4}\right)$$

$$* \quad \Psi_5(x) = 1 + \left(1 - \frac{3}{2}x\right)\left[\lambda + \frac{\lambda^3}{4} + \frac{\lambda^5}{16}\right] + \frac{\lambda^2}{4}\left[1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16}\right]$$

$\vdots$

$$* \quad \Psi_n(x) = 1 + \left(1 - \frac{3}{2}x\right)\left[\lambda + \frac{\lambda^3}{4} + \frac{\lambda^5}{4^2} + \frac{\lambda^7}{4^3} + \cdots\right] + \frac{\lambda^2}{4}\left[1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{4^2} + \frac{\lambda^6}{4^3} + \cdots\right]$$

$$\Psi(x) = \lim_{n\to\infty}\Psi_n(x) = 1 + \left(1 - \frac{3}{2}x\right)\left(\lambda + \frac{\lambda^3}{4} + \frac{\lambda^5}{4^2} + \frac{\lambda^7}{4^3} + \cdots\right] + \left[\frac{\lambda^2}{4} + \left(\frac{\lambda^2}{4}\right)^2 + \left(\frac{\lambda^2}{4}\right)^3 + \left(\frac{\lambda^2}{4}\right)^4 + \cdots\right]$$

$$= 1 + \lambda\left(1 - \frac{3}{2}x\right)\sum_{n=0}^{\infty}\left(\frac{\lambda^2}{4}\right)^n + \frac{\lambda^2}{4}\sum_{n=0}^{\infty}\left(\frac{\lambda^2}{4}\right)^n$$

وهي متسلسلة هندسية :

$$\sum_{n=0}^{\infty}\left(\frac{\lambda^2}{4}\right)^n = \frac{1}{1 - \frac{\lambda^2}{4}} = \frac{4}{4 - \lambda^2}$$

$$\Rightarrow \Psi(x) = 1 + \lambda\left(1 - \frac{3}{2}x\right)\cdot\frac{4}{4 - \lambda^2} + \frac{\lambda^2}{4}\cdot\frac{4}{4 - \lambda^2}$$

$$\Psi(x) = \frac{4 + 2\lambda(2 - 3x) + \lambda^2}{4 - \lambda^2} \quad ; \quad \lambda \neq \mp 2$$

وهو الحل المطلوب .

AL SAMRAH

Subject

طريقة النوى المكررة :

$* \; K_1(x,t) = K(x,t) = 1-3xt$

$* \; K_2(x,t) = \int_a^b K_1(x,z)\cdot K_1(z,t)\, dz = \int_0^1 (1-3xz)(1-3zt)\, dz$

$= \int_0^1 (1-3zt-3xz+9xtz^2)\, dz$

$= 1-\frac{3}{2}t-\frac{3}{2}x+3xt$

$* \; K_3(x,t) = \int_a^b K_1(x,z)\, K_2(z,t)\, dz = \int_0^1 (1-3xz)(1-\frac{3}{2}z-\frac{3}{2}t+3tz)\, dz$

$K_3 = \int_0^1 [1-\frac{3}{2}z-\frac{3}{2}t+3tz-3xz+\frac{9}{2}xz^2+\frac{9}{2}xtz-9xtz^2]\, dz$

$K_3 = [z-\frac{3}{4}z^2-\frac{3}{2}tz+\frac{3}{2}tz^2-\frac{3}{2}xz^2+\frac{3}{2}xz^3+\frac{9}{4}xtz^2-3xtz^3]_0^1$

$K_3(x,t) = 1-\frac{3}{4}-\frac{3}{2}t+\frac{3}{2}t-\frac{3}{2}x+\frac{3}{2}x+\frac{9}{4}xt-3xt$

$K_3(x,t) = \frac{1}{4}-\frac{3}{4}xt = \frac{1}{4}(1-3xt) = \frac{1}{4}K_1(x,t)$

وكذلك سنجد أنّ :

$* \; K_4(x,t) = \frac{1}{4}K_2(x,t)$

وكذلك :

$* \; K_5(x,t) = \frac{1}{4}K_3(x,t)$

$\vdots$

$* \; K_m(x,t) = \frac{1}{4}K_{m-2}(x,t)$

نعوض في متسلسلة نيومن :

$\psi(x) = 1+\lambda\int_0^1 (1-3xt)\, dt + \lambda^2\int_0^1 (1-\frac{3}{2}t-\frac{3}{2}x+3xt)\, dt +$

$+\lambda^3\int_0^1 \frac{1}{4}(1-3xt)\, dt + \lambda^4\int_0^1 \frac{1}{4}(1-\frac{3}{2}t-\frac{3}{2}x+3xt)\, dt + \cdots\cdots$

$$\Psi(x) = 1 + \int_0^1 (1-3xt)\,dt \left[\lambda + \frac{\lambda^3}{4} + \frac{\lambda^5}{4^2} + \cdots\right] + \int_0^1 (1 - \frac{3}{2}t - \frac{3}{2}xt + 3xt)\,dt \left[\lambda^2 + \frac{\lambda^4}{4} + \frac{\lambda^6}{4^2} + \cdots\right]$$

but: $\lambda + \frac{\lambda^3}{4} + \frac{\lambda^5}{4^2} + \cdots\cdots = \lambda\left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{4^2} + \cdots\right)$

$$\Rightarrow \Psi(x) = 1 + \left(1 - \frac{3}{2}x\right)\left(\frac{4\lambda}{4-\lambda^2}\right) + \frac{4\lambda^2}{4-\lambda^2}$$

$$\Rightarrow \Psi(x) = \frac{4 + 2\lambda(2-3x)}{4-\lambda^2} \quad ; \quad |\lambda| < \frac{1}{B}$$

وهو تمثل حلا لمعادلة فريدهولم المعطاة.

مثال 2: استخدم طريقة التقريبات المتتالية لحل المعادلة:

$$\Psi(x) = 1 + \int_0^1 xt^2 \cdot \Psi(t)\,dt$$

$* \int_0^1 |h(x)|^2\,dt = \int_0^1 dt = 1 < \infty$

$* \int_0^1 |k(x,t)|^2\,dx = \int_0^1 (xt^2)^2\,dx = \int_0^1 x^2t^4\,dx = \left[\frac{x^3}{3}t^4\right]_0^1 = \frac{t^4}{3} < \infty \;;\; 0<t<1.$

$* \int_0^1 |k(x,t)|^2\,dt = \int_0^1 x^2t^4\,dt = \left[x^2\frac{t^5}{5}\right]_0^1 = \frac{x^2}{5} < \infty \;;\; 0<x<1$

$* \int_0^1\int_0^1 |k(x,t)|^2\,dx\,dt = \int_0^1\int_0^1 x^2t^4\,dx\,dt = \int_0^1 \frac{t^4}{3}\,dt = \left[\frac{t^5}{15}\right]_0^1 = \frac{1}{15} < \infty$

$* \Psi_0(x) = h(x) = 1$

$* \Psi_1(x) = h(x) + \lambda\int_a^b k(x,t)\,\Psi_0(t)\,dt = 1 + \left[\frac{xt^3}{3}\right]_0^1 = 1 + \frac{x}{3}$

$* \Psi_2(x) = h(x) + \lambda\int_a^b k(x,t)\cdot\Psi_1(t)\,dt = 1 + \int_0^1 xt^2\left(1 + \frac{t}{3}\right)dt$

$$\Psi_2(x) = 1 + \frac{x}{3} + \frac{x}{3.4}$$

Subject

* $\Psi_3(x) = h(x) + \lambda \int_a^b k(x,t) \cdot \Psi_2(t)\, dt = 1 + \int_0^1 x t^2 \left(1 + \frac{t}{3} + \frac{t}{3.4}\right) dt$

$$\Psi_3(x) = 1 + \frac{x}{3} + \frac{x}{3.4} + \frac{x}{3.4^2}$$

$\vdots$

* $\Psi_n(x) = 1 + \frac{x}{3} + \frac{x}{3.4} + \frac{x}{3.4^2} + \frac{x}{3.4^3} + \cdots + \frac{x}{3.4^{n-1}}$

$$\Psi(x) = \lim_{n \to \infty} \Psi_n(x) = 1 + \frac{x}{3}\left[1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right]$$

$$\Psi(x) = 1 + \frac{x}{3} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1}$$

$$\Psi(x) = 1 + \frac{x}{3}\left[\frac{1}{1-\frac{1}{4}}\right] = 1 + \frac{4}{9}x$$

وهو حل المعادلة المعطاة .

حل المعادلة التكاملية الآتية :

$$\Psi(x) = \cot x + \lambda \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} t \cdot \tan t \cdot \Psi(t)\, dt$$

مستخدماً التقريبات المتتالية .