rows of $A$. A matrix in $\mathbb{C}^{n\times m}$ is of *full rank* when its rank is equal to the smallest of $m$ and $n$. A fundamental result of linear algebra is stated by the following relation:

$$\mathbb{C}^n = \operatorname{Ran}(A) \oplus \operatorname{Ker}(A^T). \tag{1.17}$$

The same result applied to the transpose of $A$ yields $\mathbb{C}^m = \operatorname{Ran}(A^T) \oplus \operatorname{Ker}(A)$.

A subspace $S$ is said to be *invariant* under a (square) matrix $A$ whenever $AS \subset S$. In particular, for any eigenvalue $\lambda$ of $A$ the subspace $\operatorname{Ker}(A - \lambda I)$ is invariant under $A$. The subspace $\operatorname{Ker}(A - \lambda I)$ is called the eigenspace associated with $\lambda$ and consists of all the eigenvectors of $A$ associated with $\lambda$, in addition to the zero vector.

## 1.7 Orthogonal Vectors and Subspaces

A set of vectors $G = \{a_1, a_2, \ldots, a_r\}$ is said to be *orthogonal* if

$$(a_i, a_j) = 0 \quad \text{when} \quad i \neq j.$$

It is *orthonormal* if, in addition, every vector of $G$ has a 2-norm equal to unity. A vector that is orthogonal to all the vectors of a subspace $S$ is said to be orthogonal to this subspace. The set of all the vectors that are orthogonal to $S$ is a vector subspace called the *orthogonal complement* of $S$ and denoted by $S^\perp$. The space $\mathbb{C}^n$ is the direct sum of $S$ and its orthogonal complement. Thus, any vector $x$ can be written in a unique fashion as the sum of a vector in $S$ and a vector in $S^\perp$. The operator that maps $x$ into its component in the subspace $S$ is the *orthogonal projector* onto $S$.

Every subspace admits an orthonormal basis that is obtained by taking any basis and *orthonormalizing* it. The orthonormalization can be achieved by an algorithm known as the Gram–Schmidt process, which we now describe.

Given a set of linearly independent vectors $\{x_1, x_2, \ldots, x_r\}$, first normalize the vector $x_1$, which means divide it by its 2-norm, to obtain the scaled vector $q_1$ of norm unity. Then $x_2$ is orthogonalized against the vector $q_1$ by subtracting from $x_2$ a multiple of $q_1$ to make the resulting vector orthogonal to $q_1$; i.e.,

$$x_2 \leftarrow x_2 - (x_2, q_1)q_1.$$

The resulting vector is again normalized to yield the second vector $q_2$. The $i$th step of the Gram–Schmidt process consists of orthogonalizing the vector $x_i$ against all previous vectors $q_j$.

**ALGORITHM 1.1. Gram–Schmidt**

1. *Compute* $r_{11} := \|x_1\|_2$. *If* $r_{11} = 0$ *Stop, else compute* $q_1 := x_1/r_{11}$
2. *For* $j = 2, \ldots, r$, *Do*
3.     *Compute* $r_{ij} := (x_j, q_i)$ *for* $i = 1, 2, \ldots, j-1$
4.     $\hat{q} := x_j - \sum_{i=1}^{j-1} r_{ij} q_i$
5.     $r_{jj} := \|\hat{q}\|_2$
6.     *If* $r_{jj} = 0$ *then Stop, else* $q_j := \hat{q}/r_{jj}$
7. *EndDo*

It is easy to prove that the above algorithm will not break down; i.e., all $r$ steps will be completed iff the set of vectors $x_1, x_2, \ldots, x_r$ is linearly independent. From lines 4 and 5, it is clear that at every step of the algorithm the following relation holds:

$$x_j = \sum_{i=1}^{j} r_{ij} q_i.$$

If $X = [x_1, x_2, \ldots, x_r]$, $Q = [q_1, q_2, \ldots, q_r]$, and $R$ denotes the $r \times r$ upper triangular matrix whose nonzero elements are the $r_{ij}$'s defined in the algorithm, then the above relation can be written as

$$X = QR. \tag{1.18}$$

This is called the QR decomposition of the $n \times r$ matrix $X$. From what was said above, the QR decomposition of a matrix exists whenever the column vectors of $X$ form a linearly independent set of vectors.

The above algorithm is the standard Gram–Schmidt process. There are alternative formulations of the algorithm that have better numerical properties. The best known of these is the modified Gram–Schmidt (MGS) algorithm.

**ALGORITHM 1.2. MGS**

1. *Define* $r_{11} := \|x_1\|_2$. *If* $r_{11} = 0$ *Stop, else* $q_1 := x_1 / r_{11}$
2. *For* $j = 2, \ldots, r$, *Do*
3.     *Define* $\hat{q} := x_j$
4.     *For* $i = 1, \ldots, j-1$, *Do*
5.         $r_{ij} := (\hat{q}, q_i)$
6.         $\hat{q} := \hat{q} - r_{ij} q_i$
7.     *EndDo*
8.     *Compute* $r_{jj} := \|\hat{q}\|_2$
9.     *If* $r_{jj} = 0$ *then Stop, else* $q_j := \hat{q} / r_{jj}$
10. *EndDo*

Yet another alternative for orthogonalizing a sequence of vectors is the Householder algorithm. This technique uses Householder *reflectors*, i.e., matrices of the form

$$P = I - 2ww^T, \tag{1.19}$$

in which $w$ is a vector of 2-norm unity. Geometrically, the vector $Px$ represents a mirror image of $x$ with respect to the hyperplane $\text{span}\{w\}^\perp$.

To describe the Householder orthogonalization process, the problem can be formulated as that of finding a QR factorization of a given $n \times m$ matrix $X$. For any vector $x$, the vector $w$ for the Householder transformation (1.19) is selected in such a way that

$$Px = \alpha e_1,$$

where $\alpha$ is a scalar. Writing $(I - 2ww^T)x = \alpha e_1$ yields

$$2w^T x \, w = x - \alpha e_1. \tag{1.20}$$

المحاضرة السادسة:
"تتمة المصفوفة المتعامدة":
خوارزمية غرام - شميدت

1- Compute $r_{11} := \|x_1\|_2$

إذا كانت:
$r_{11} = 0$

توقف

وإلا:
Compute $q_1 := x_1 / r_{11}$

2- for $j = 2, \ldots, r$ Do
3- Compute $r_{ij} := (x_j, q_i)$ for $i = 1, 2, \ldots, j-1$
4- $\hat{q} = x_j - \sum_{i=1}^{j-1} r_{ij} q_i$
5- $r_{jj} := \|\hat{q}\|_2$
6- If $r_{jj} = 0$ then stop, else $q_j := \hat{q} / r_{jj}$
7- EndDo

"الصيغة العامة لها"
من السهل أن نثبت أن الخوارزمية السابقة لن تتوقف
تقريباً، كل الخطوات، سيتم إذا وفقط إذا كانت الأشعة $x_1, x_2, \ldots x_r$
مستقلة خطياً.
من السطرين 4 و 5، واضح أنه في كل خطوة من الخوارزمية تتحقق العلاقة
التالية

$$x_j = \sum_{i=1}^{j} r_{ij} q_i.$$

إذا كانت $X = [x_1, x_2, \ldots x_r]$ ، $Q_r = [q_1, q_2, \ldots q_r]$ وكانت $R$ مصفوفة
مثلثية عليا من المرتبة $r \times r$ والتي عناصرها الغير صفرية هي عبارة عن $r_{ij}$
المعرفة في الخوارزمية. عندها العلاقة السابقة يمكن كتابتها بالشكل:

$$X = QR \qquad (1.18)$$

إن هذا يدعى بالتحليل QR للمصفوفة X من المرتبة $n \times 2$،
نستنتج مما سبق سابقاً أن QR للمصفوفة موجود فيما لو كانت أعمدة
المصفوفة تشكل مجموعة مستقلة خطياً من الأشعة.
إن الخوارزمية السابقة هي طريقة غرام شميت الكلاسيكية.
إن هناك صياغات بديلة للخوارزمية التي تمتلك خواص عددية أفضل.
أشهرها هي خوارزمية غرام شميت المعدلة.

ALGORITHM 1.2. MGS غ معدلة

حيث الآن هناك بديل آخر لتشكيل متتالية من الأشعة بدل خوارزمية [illegible]
هذه التقنية تستخدم عاكسات هاوس هولدر
وهي مصفوفات من الشكل:

$$P = I - 2ww^T \qquad (1.19)$$

حيث w هو شعاع ذو نظيم إقليدي واحدي.
هندسياً، الشعاع Px يمثل الصورة المعاكسة لـ x المتعلقة بالفضاء المعامد
الفضاء المولد بـ w.
لو صف تحليل تشكل هاوس هولدر، يمكن أن نضع المسألة إلى إيجاد QR
للمصفوفة المعطاة X من المرتبة $n \times m$
من أجل أي شعاع x، الشعاع w

$$Px = \alpha e_1$$

where $\alpha$ is a scalar. writing $(I - 2ww^T)x = \alpha e_1$ yields

$$2w^T x w = x - \alpha e_1$$

- السنة الجامعية السادسة -