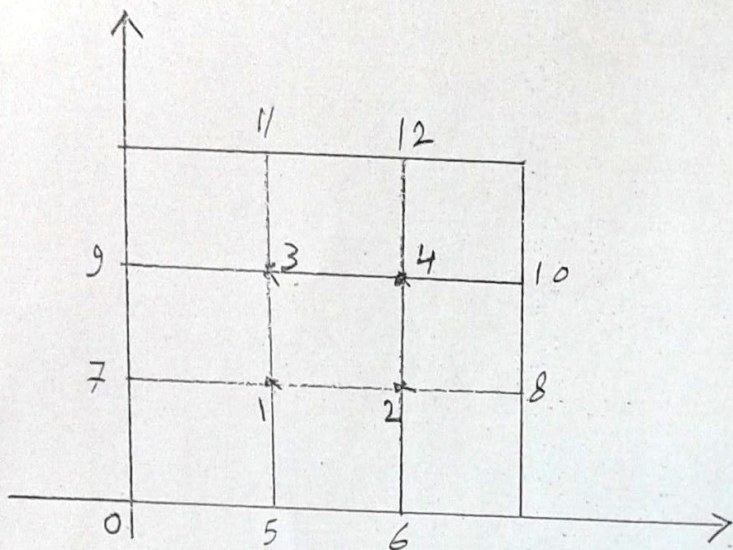


$u(x,y) = x^2 - y^2$ on ∂D في هذه الحالة $\nabla^2 u = 0$ ليكن لدينا

$0 \leq y \leq 1$, $0 \leq x \leq 1$ حيث D مربع



الحل : $k = h = \frac{1}{3}$

$$u_5 = u\left(\frac{1}{3}, 0\right) = \frac{1}{9} \quad , \quad u_6 = u\left(\frac{2}{3}, 0\right) = \frac{4}{9} \quad , \quad u_7 = u\left(0, \frac{1}{3}\right) = -\frac{1}{9}$$

$$u_8 = u\left(1, \frac{1}{3}\right) = \frac{8}{9} \quad , \quad u_9 = u\left(0, \frac{2}{3}\right) = -\frac{4}{9} \quad , \quad u_{10} = u\left(1, \frac{2}{3}\right) = \frac{5}{9}$$

$$u_{11} = u\left(\frac{1}{3}, 1\right) = -\frac{8}{9} \quad , \quad u_{12} = u\left(\frac{2}{3}, 1\right) = -\frac{5}{9}$$

رتبنا المعادلات التي حصلنا عليها في الجوارد (بصفة بي)

$$u_2 + u_3 + u_7 + u_5 - 4u_1 = 0 \Rightarrow -4u_1 + u_2 + u_3 = 0$$

$$u_9 + u_4 + u_1 + u_6 - 4u_2 = 0 \Rightarrow u_1 - 4u_2 + u_4 = -\frac{4}{3}$$

$$u_4 + u_{11} + u_9 + u_7 - 4u_3 = 0 \Rightarrow u_1 - 4u_3 + u_4 = \frac{4}{3}$$

$$u_{10} + u_{12} + u_3 + u_2 - 4u_4 = 0 \Rightarrow u_2 + u_3 - 4u_4 = 0$$

$$A = \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ -\frac{4}{3} \\ \frac{4}{3} \\ 0 \end{bmatrix} \left\{ \begin{array}{l} u_1 = 0 \\ u_2 = \frac{1}{3} \\ u_3 = -\frac{1}{3} \\ u_4 = 0 \end{array} \right.$$

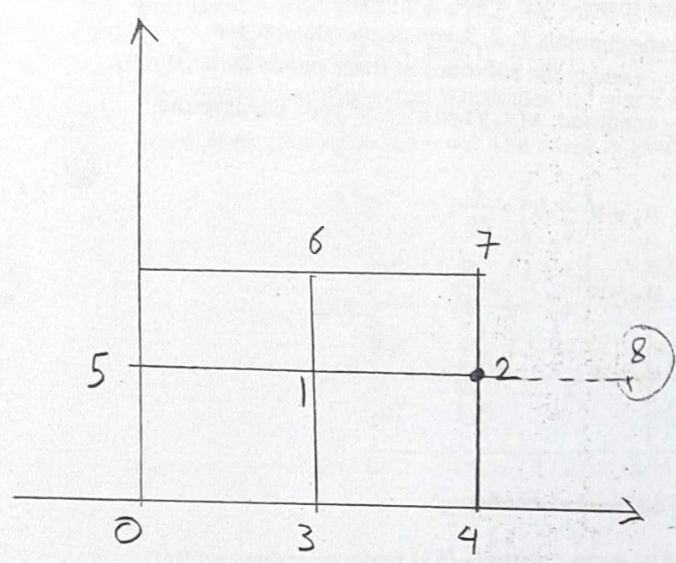
مثال:

ارصد كل $\nabla^2 u = 0$ في D حيث D هي مربع $0 \leq x \leq 1, 0 \leq y \leq 1$

$u(x, y) = x^2 - y^2$ on $x=0, y=0, y=1$

$u + \frac{\partial u}{\partial x} = x^2 + 2x - y^2$ on $x=1$

الحل: سنأخذ $h = k = \frac{1}{2}$



$u_3 = u(\frac{1}{2}, 0) = \frac{1}{4}$
 $u_4 = u(1, 0) = 1$
 $u_5(0, \frac{1}{2}) = -\frac{1}{4}$
 $u_6(\frac{1}{2}, 1) = -\frac{3}{4}$
 $u_7(1, 1) = 0$

الشرط الطبيعي

$u_2 + u_6 + u_5 + u_3 - 4u_1 = 0$

$u_2 - \frac{3}{4} - \frac{1}{4} + \frac{1}{4} - 4u_1 = 0$

① $-4u_1 + u_2 = \frac{3}{4}$

بالتعريف في المعادلات عن المحاور كل نقطة

$u + \frac{\partial u}{\partial x} = u_2 + \frac{1}{2h} [u_8 - u_1]$ $u_2(1, \frac{1}{2})$ وبالاستعانة بالنقطة

$= u_2 + u_8 - u_1$
 $= 1 + 2 - \frac{1}{4} = \frac{11}{4}$

بالتعريف بالمعادلات عند النقطة u_2

$u_8 + u_7 + u_1 + u_4 - 4u_2 = 0$

$u_1 - u_2 + \frac{11}{4} + 0 + u_1 + 1 - 4u_2 = 0$

② $2u_1 - 5u_2 = -\frac{15}{4}$ \Rightarrow $u_1 = 0, u_2 = \frac{3}{4}$

Example 3: Find the solution of $\nabla^2 u = x + y$ in R subject to the boundary conditions $u(x, y) = (x^3 + y^3)/6$ on Γ , where R is the triangle $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1$, using the five point formula. Assume uniform step length $h = \frac{1}{4}$ along the axes.

Solution: The boundaries of the triangle are $x = 0, y = 0$ and $x + y = 1$. The mesh is given in Fig.9. We have three mesh points 1, 2, 3 whose coordinates are $(1/4, 1/4), (1/2, 1/4), (1/4, 1/2)$. Denote the solutions at these points as u_1, u_2, u_3 respectively. From the boundary condition, $u(x, y) = (x^3 + y^3)/6$, we have the following values.

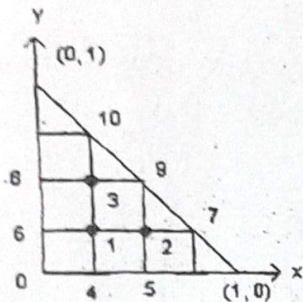


Fig.9: Mesh in Example 3

$$u_4 = u\left(\frac{1}{4}, 0\right) = \frac{1}{384}, \quad u_5 = u\left(\frac{1}{2}, 0\right) = \frac{1}{48},$$

$$u_6 = u\left(0, \frac{1}{4}\right) = \frac{1}{384}, \quad u_7 = u\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{7}{96},$$

$$u_8 = u\left(0, \frac{1}{2}\right) = \frac{1}{48}, \quad u_9 = u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{24},$$

$$u_{10} = u\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{7}{96}.$$

Hence, we obtain the following difference equations

$$\text{At 1: } u_2 + u_3 + u_6 + u_4 - 4u_1 = \frac{1}{16} \left(\frac{1}{4} + \frac{1}{4} \right)$$

$$\text{or, } -4u_1 + u_2 + u_3 = \frac{1}{32} - \frac{1}{384} - \frac{1}{384} = \frac{5}{192}.$$

$$\text{At 2: } u_7 + u_9 + u_1 + u_5 - 4u_2 = \frac{1}{16} \left(\frac{1}{2} + \frac{1}{4} \right)$$

$$\text{or, } u_1 - 4u_2 = \frac{3}{64} - \frac{7}{96} - \frac{1}{24} - \frac{1}{48} = -\frac{34}{384}.$$

$$\text{At 3: } u_9 + u_{10} + u_8 + u_1 - 4u_3 = \frac{1}{16} \left(\frac{1}{4} + \frac{1}{2} \right)$$

$$\text{or, } u_1 - 4u_3 = \frac{3}{64} - \frac{1}{24} - \frac{7}{96} - \frac{1}{48} = -\frac{34}{384}.$$

Subtracting the last two equations, we get $u_2 = u_3$. Therefore, we have the equations

$$-4u_1 + 2u_2 = \frac{5}{192}, \quad u_1 - 4u_2 = -\frac{34}{384},$$

whose solution is $u_1 = 1/192, u_2 = u_3 = 9/384$.

You may now try the following exercises.

Example 7: Find the solution of the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1$$

$$u(x, 0) = \sin(\pi x), \quad 0 \leq x \leq 1; \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 \leq x \leq 1,$$

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0.$$

by using the (i) explicit scheme (48) and (ii) the implicit scheme (52). Use the central difference approximation to the derivative to obtain initial condition. Assume $h = 1/4$, $r = 1/2$. Integrate for one time step. Compare with the exact solution $u(x, t) = \sin(\pi x) \cos(\pi t)$.

Solution: For $h = 1/4$, $r = 1/2$, we have $k = rh = 1/8$. The nodal points are given by $x_i = ih$, $i = 0, 1, 2, 3, 4$, and $t_n = nk$, $n = 0, 1, 2, \dots$

The initial conditions give the values $u_i^0 = \sin(\pi i/4)$, $i = 0, 1, 2, 3, 4$. The boundary conditions give $u_i^n = 0$, $i = 0, 4$. The mesh is given in Fig. 19.

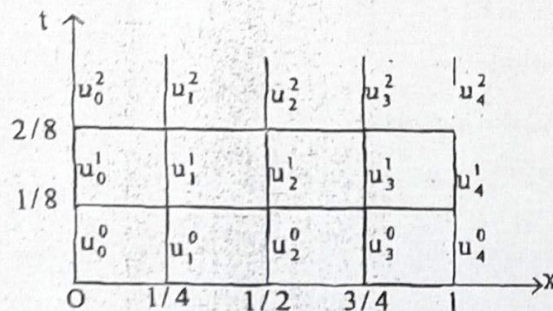


Fig. 19: Mesh in Example 7

(i) The explicit method (48) gives

$$u_i^{n+1} = \frac{3}{2}u_i^n + \frac{1}{4}(u_{i-1}^n + u_{i+1}^n) - u_i^{n-1}, \quad i = 1, 2, 3.$$

Since, $u_i(x, 0) = 0$, we get $u_i^{-1} = u_i^1$, $i = 1, 2, 3$.

For $n = 0$, we get

$$u_i^1 = \frac{3}{2}u_i^0 + \frac{1}{4}(u_{i-1}^0 + u_{i+1}^0) - u_i^{-1}$$

$$\text{or, } u_i^1 = \frac{3}{4}u_i^0 + \frac{1}{8}(u_{i-1}^0 + u_{i+1}^0).$$

Substituting the initial values, we get

$$u_1^1 = \frac{3}{4}u_1^0 + \frac{1}{8}(u_0^0 + u_2^0) = \frac{3}{4}\sin\left(\frac{\pi}{4}\right) + \frac{1}{8}(0 + 1) = 0.65533,$$

$$u_2^1 = \frac{3}{4}u_2^0 + \frac{1}{8}(u_1^0 + u_3^0) = \frac{3}{4} + \frac{1}{8}\left[\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right)\right] = 0.92678,$$

$$u_3^1 = \frac{3}{4}u_3^0 + \frac{1}{8}(u_2^0 + u_4^0) = \frac{3}{4}\sin\left(\frac{3\pi}{4}\right) + \frac{1}{8}(1 + 0) = 0.65593.$$

we have $t_1 = k = 1/8$.

The exact solution is

$$u\left(\frac{1}{4}, \frac{1}{8}\right) = u\left(\frac{3}{4}, \frac{1}{8}\right) = 0.65328, \quad u\left(\frac{1}{2}, \frac{1}{8}\right) = 0.92388.$$

(ii) The implicit method is given by

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + \frac{\tau^2}{2} \delta_x^2 [u_i^{n+1} + u_i^{n-1}]$$

$$\text{or, } -\frac{1}{8}u_{i-1}^{n+1} + \frac{5}{4}u_i^{n+1} - \frac{1}{8}u_{i+1}^{n+1} = 2u_i^n + \frac{1}{8}u_{i-1}^{n-1} - \frac{5}{4}u_i^{n-1} + \frac{1}{8}u_{i+1}^{n-1}.$$

From the initial condition $u_i(x, 0) = 0$, we get $u_i^{-1} = u_i^1$. Hence, we get

$$-\frac{1}{4}u_{i-1}^1 + \frac{10}{4}u_i^1 - \frac{1}{4}u_{i+1}^1 = 2u_i^0.$$

For $i = 1, 2, 3$, we get the system of equations

$$-\frac{1}{4}u_0^1 + \frac{10}{4}u_1^1 - \frac{1}{4}u_2^1 = 2u_1^0 = 2\sin(\pi/4) = \sqrt{2}$$

$$-\frac{1}{4}u_1^1 + \frac{10}{4}u_2^1 - \frac{1}{4}u_3^1 = 2u_2^0 = 2\sin(\pi/2) = 2$$

$$-\frac{1}{4}u_2^1 + \frac{10}{4}u_3^1 - \frac{1}{4}u_4^1 = 2u_3^0 = 2\sin\left(\frac{3\pi}{4}\right) = \sqrt{2}.$$

Substituting $u_0^1 = 0, u_4^1 = 0$, we get

$$\begin{bmatrix} 10 & -1 & 0 \\ -1 & 10 & -1 \\ 0 & -1 & 10 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} \\ 8 \\ 4\sqrt{2} \end{bmatrix}$$

The solution of this system is (solve by Gauss elimination)

$$u_1^1 = u_3^1 = 0.65886, \quad u_2^1 = 0.93177.$$

Try the following exercises now.

E14) Show that the method (51) is of order $k^2 + h^2$.

E15) Solve Example 7, with $h = 1/4$ and $\tau = 1/3$. Integrate for one time step. Compare with the exact solution $u(x, t) = \sin(\pi x) \cos(\pi t)$.