لتكن $f: R^2 \to R$

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{x^2y + xy^2}{x^2+y^2} \sin(x-y) & (x,y) \neq (0,0) \end{cases}$$

اثبت أن

$f_{xy}(0,0) = -1 \quad , \quad f_{yx}(0,0) = 1$

$f_{yy}(0,0) = 0 \quad , \quad f_{xx}(0,0) = 0 \quad , f_y(0,0) = 0 \quad , f_x(0,0) = 0$

الحل:

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_x(x,y) = \frac{(2xy+y^2)(x^2+y^2) - 2x(x^2y+xy^2)}{(x^2+y^2)^2} \sin(x-y) + \frac{x^2y + xy^2}{x^2+y^2} \cdot \cos(x-y)$$

$$f_x(x,y) = \frac{-x^2y^2 + 2xy^3 + y^4}{(x^2+y^2)^2} \sin(x-y) + \frac{x^2y + xy^2}{x^2+y^2} \cos(x-y) \qquad (x,y) \neq (0,0)$$

وعندما يكون $(x,y) = (0,0)$ فتكون $f_x(x,y) = 0$

$$f_y(x,y)=\begin{cases} 0 & (x,y)=(0,0) : \text{بنفس الطريقة} \\ \dfrac{-x^2y^2+2\cdot y\cancel{x}^3+\cancel{x}^4}{(x^2+y^2)^2}\sin(x-y)-\dfrac{x^2y+xy^2}{x^2+y^2}\cos(x-y) & : (x,y)\neq(0,0) \end{cases}$$

$\star\ f_{xx}(0,0)=\lim\limits_{h\to 0}\dfrac{f(h,0)-f_x(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{0-0}{h}=0$

$\star\ f_{yy}(0,0)=\lim\limits_{h\to 0}\dfrac{f_y(0,h)-f_y(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{0-0}{h}=0$

$\star\ f_{xy}(0,0)=\dfrac{\partial^2 f}{\partial y\,\partial x}(0,0)=\lim\limits_{h\to 0}\dfrac{f_x(0,h)-f_x(0,0)}{h}$

$=\lim\limits_{h\to 0}\dfrac{-\frac{h^4}{h^4}\sin h}{h}=\lim\limits_{h\to 0}\dfrac{-\sin h}{h}=-1$

$\star\ f_{yx}(0,0)=(f_y)_x=\dfrac{\partial}{\partial x}f_y=\lim\limits_{h\to 0}\dfrac{f_y(h,0)-f_y(0,0)}{h}$

$=\lim\limits_{h\to 0}\dfrac{\sin h}{h}=1$

تمرين: لتكن $f:\mathbb{R}^2\to\mathbb{R}$ (دالة معطاة من المحاضرة السابقة)

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^6+2y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

بيّن أنه يوجد دالة مشتق اتجاهي f باتجاه $u(\alpha,\beta)$ حيث $\alpha^2+\beta^2=1$ في النقطة $(0,0)$ في حين أن f غير مستمرة عند $(0,0)$

$\|u\| = 1$

الحل:

$$\frac{\partial F}{\partial u}(0,0) = \lim_{h\to 0} \frac{f(c+hu) - f(c)}{h}$$

$$c + hu = (0,0) + h(\alpha,\beta) = (h\alpha, h\beta)$$

$$\frac{\partial F}{\partial u}(0,0) = \lim_{h\to 0} \frac{\frac{h^3\alpha^2\beta}{h^6\alpha^6 + 2h^2\beta^2} - 0}{h}$$

$$\lim_{h\to 0} \frac{\alpha^2\beta}{h^4\alpha^6 + 2\beta^2} = \frac{\alpha^2}{2\beta}\beta \neq 0$$

طريقة المتتاليات:

نأخذ متتالية

$$\left\{\frac{1}{n}, \frac{1}{n^2}\right\} \xrightarrow[n\to\infty]{} (0,0)$$

$$\lim_{n\to\infty}\left(\frac{1}{n}, \frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^6} + 2\frac{1}{n^4}} = \lim_{n\to\infty} \frac{1}{\frac{1}{n^2}+2}$$

$$= \lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$$

$$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n^2}\right) \neq f(0,0) \Longrightarrow$$ إذن f غير مستمر عند $(0,0)$

أريد أن أطابق التعريف

$$\forall \varepsilon > 0, \exists \delta > 0, \|(x,y) - (0,0)\| < \delta \Rightarrow \|f(x,y) - f(0,0)\| < \varepsilon$$

ليكن $x = \frac{\delta}{2}$, $y = \frac{\delta^2}{4}$

$$x^2 + y^2 = \frac{\delta^2}{4} + \frac{\delta^4}{16} = \frac{\delta^2}{4}\left(1 + \frac{\delta^2}{4}\right) < \delta^2$$

$1+\frac{\delta^2}{4} < 4 \Rightarrow \frac{\delta^2}{4} < 3 \to \delta^2 < 12 \Rightarrow \delta < 2\sqrt{3}$

$x^2+y^2 = \frac{\delta^2}{4} + \frac{\delta^4}{16} = \frac{\delta^2}{4}\left(1+\frac{\delta^2}{4}\right) < \delta^2$ ومنه

$(\delta < 2\sqrt{3})$

$\left|f(x,y)-f(0,0)\right| = \left| \frac{\frac{\delta^2}{4}\cdot\frac{\delta^2}{4}}{\frac{\delta^6}{\frac{6}{2}} + \frac{2\delta^4}{16}} - 0 \right| = \left| \frac{1}{\frac{\delta^2}{4}+2} \right|$

$\left|\frac{1}{\frac{\delta^2}{4}+2}\right| > \frac{1}{5} > 0 \Leftarrow$ ومنه $\frac{\delta^2}{4}+2 < 5$ إذن

تمرين: لتكن $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$

$f(x) = \|x\|^{2-n} \qquad n \geq 3$

اثبت أن: $\sum_{i=1}^{n} \frac{\partial^2 F}{\partial x^2}(x) = 0$

الحل:

$f(x) = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{2-n}{2}} \quad \frac{2-n}{2} = 1-\frac{n}{2}$

$\frac{\partial F}{\partial x_1} = \left(1-\frac{n}{2}\right)(2x_1)(x_1^2+x_2^2+\dots x_n)^{-\frac{n}{2}}$

$\frac{\partial^2 F}{\partial x_1^2} = \left(1-\frac{n}{2}\right)\left(-\frac{n}{2}\right)(2x_1)^2(x_1^2+x_2^2+\dots+x_n^2)^{-\frac{n}{2}-1}$

$+ \left(1-\frac{n}{2}\right)(2)(x_1^2+x_2^2+\dots+x_n^2)^{-\frac{n}{2}}$

$= 2\left(1-\frac{n}{2}\right)(x_1^2+x_2^2+\dots+x_n^2)^{-\frac{n}{2}-1}\left[-nx_1^2 + (x_1^2 + \dots x_n^2)\right]$

$$\frac{\partial^2 F}{\partial x_2^2} = 2\left(1-\frac{n}{2}\right)\left(x_1^2+x_2^2+\dots+x_n^2\right)^{-\frac{n}{2}-1} \cdot \left[-n x_2^2 + \left(x_1^2+x_2^2+\dots+x_n^2\right)\right]$$

$$\sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i^2} = 2\left(1-\frac{n}{2}\right)\left(x_1^2+x_2^2+\dots+x_n^2\right)^{-\frac{n}{2}-1} \begin{bmatrix} -n\cancel{(x_1^2+\dots+x_n^2)} \\ +n\cancel{(x_1^2+\dots+x_n^2)} \end{bmatrix}$$

ومنه نجد أن

$$\sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i^2} = 0$$

لتكن $f : R^2 \to R$

$$f(x,y) = x^{\frac{1}{3}} \cdot y^{\frac{2}{3}}$$

حدد من بين المشتقات ما كان موجوداً منها واحسب في حال وجود المشتق قيمته العددية

$$\frac{\partial f}{\partial x}(0,1), \quad \frac{\partial f}{\partial x}(0,0), \quad \frac{\partial f}{\partial e}(0,0), \quad \frac{\partial f}{\partial e}(0,1)$$

$$e = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

الحل

$$\star\ f_x(0,0) = \frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\star\ f_x(0,1) = \lim_{h\to 0} \frac{f(h,1)-f(0,1)}{h} = \lim_{h\to 0} \frac{h^{\frac{1}{3}}-0}{h} = \infty$$

$$\star\ \frac{\partial f}{\partial e}(0,0) = \lim \frac{f(c+he)-f(c)}{h} = \lim_{h\to 0} \frac{f\left(h\frac{\sqrt{2}}{2}, h\frac{\sqrt{2}}{2}\right)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h\frac{\sqrt{2}}{2}-0}{h} = \frac{\sqrt{2}}{2}$$

$* \frac{\partial f}{\partial e}(0,1) = \lim_{h\to 0} \frac{f(c+he)-f(0,1)}{h}$

$= \lim_{h\to 0} \frac{f\left(\frac{\sqrt{2}}{2}h, \frac{\sqrt{2}}{2}h+1\right)-f(0,1)}{h} = \lim_{h\to 0} \frac{\left(\frac{\sqrt{2}}{2}h\right)^{\frac{1}{3}}\left(\frac{\sqrt{2}}{2}h+1\right)^{\frac{2}{3}}}{h} = 0$

$\lim_{h\to 0} \frac{\left(\frac{\sqrt{2}}{2}\right)^{\frac{1}{3}}\left(\frac{\sqrt{2}}{2}h+1\right)^{\frac{2}{3}}}{h^{\frac{2}{3}}} = \lim_{h\to 0}\left(\frac{\sqrt{2}}{2}\right)^{\frac{1}{3}}\left(\frac{\frac{\sqrt{2}}{2}h+1}{h}\right)^{\frac{2}{3}}$

$= \infty$

وظيفة: $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} x^2 \operatorname{arctg} \frac{y}{x} - y^2 \operatorname{Arctg} \frac{x}{y} & (x,y) \neq (0,0) \\ 0 & y=0 \text{ أو } x=0 \end{cases}$$

نعتبر من هنا أن $|\operatorname{Arctg} u| < \frac{\pi}{2}$

اثبت أن $f_{xx}(0,0) = f_{yy}(0,0) = 0$

$f_{xy}(0,0) = -1 \quad , f_{yx}(0,0) = 1$

انتهت المحاضرة