Limits of functions

Limit of a function. Some remarkable limits. Infinitesimal and infinite values. Finite limit. Infinite limit. Notion of an infinity.

Limit of a function. *A number L is called a limit of a function $y = f(x)$ as x tends a :*

$$\lim_{x \to a} f(x) = L,$$

if and only if for any $\varepsilon > 0$ one can find such a positive number $\delta = \delta(\varepsilon)$, depending on ε, that $|x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$.

This definition means, that L is a *limit* of a function $y = f(x)$ if the function value approaches unrestrictedly to L, when the argument value x approaches a. Geometrically it means, that for any $\varepsilon > 0$ it is possible to find such a number δ, that if x is within an interval $(a - \delta, a + \delta)$, then a function value is within an interval $(L - \varepsilon, L + \varepsilon)$. Note, that according to this definition, a function argument x only approaches a, not adopting this value! It must be considered at calculating limit of any function at a point of its discontinuity (i.e., where this function doesn't exist).

E x a m p l e . Find :

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}.$$

S o l u t i o n . Substituting $x = 3$ into the expression $\frac{x^2 - 9}{x - 3}$ we'll receive a meaningless expression $\frac{0}{0}$ (see "About meaningless expressions" in the section "Powers and roots" of the part "Algebra"). Therefore we'll solve in a different way:

$$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3$$

Here the fraction canceling is valid, because $x \neq 3$, it only approaches 3.
Now we have:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} (x + 3) = 6,$$

because, if x approaches 3 , then $x + 3$ approaches 6 .

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Some remarkable limits.

$$\lim_{x\to 0}\frac{\sin x}{x}=1;\qquad \lim_{x\to 0}\frac{\tan x}{x}=1;\qquad \lim_{x\to 0}(1+x)^{1/x}=e.$$

Infinitesimal and infinite value. If limit of some variable is equal to 0, this variable is called an *infinitesimal*.

E x a m p l e . The function $y=\sqrt{x+5}-3$ is an infinitesimal, if x approaches 4, because $\lim_{x\to 4}\sqrt{x+5}-3=0$.

If an absolute value of some variable increases unboundedly, then this variable is called an *infinite value*.

E x a m p l e . The function $\dfrac{x+1}{x-3}$ is an infinite value, if x approaches 3.

An infinite value has no a *finite* limit, but it has so called an *infinite* limit; this fact is written as:

$$\lim_{x\to 3}\frac{x+1}{x-3}=\infty.$$

The symbol ∞("infinity") doesn't mean some number, it means only that the fraction increase unboundedly if x approaches 3 . It should be noted, that the fraction can be both positive (at $x>3$) and negative (at $x<3$). If an infinite value can be only positive at any values of x, it is marked in a record. For example, at $x\to 0$ the function $y=x^{-2}$ is an infinite value, but it is positive both at $x>0$ and at $x<0$; this is expressed as:

$$\lim_{x\to 0}x^{-2}=+\infty$$

On the contrary, the function $y=-x^{-2}$ is always negative, therefore

$$\lim_{x\to 0}(-x^{-2})=-\infty.$$

According to this, a result of our example can be written as:

$$\lim_{x\to 3}\frac{x+1}{x-3}=\pm\infty.$$

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نهايات الدوال

نهاية دالة. بعض نهايات الشهرة. قيم لا متناهية في الصفر وقيم لا متناهية (لا متناهية في الكبر). نهاية محدودة. نهاية غير محدودة. مفهوم لا نهاية.

نهاية دالة: يدعى العدد L نهاية الدالة $y=f(x)$ عندما x تسعى لـ a و نكتب: $\lim_{x \to a} f(x) = L$

إذا وفقط إذا يوجد لكل $\varepsilon > 0$ عدد موجب مثل $\delta = \delta(\varepsilon)$ متعلق بـ ε بحيث $|x-a| < \delta$ يؤدي إلى أن $|f(x)-L| < \varepsilon$.

هذا التعريف يعني أن L هي نهاية دالة $y = f(x)$ إذا كانت قيمة الدالة تسعى بلا قيود إلى L وذلك عندما قيمة المتغير x يسعى إلى a.

هندسياً يعني، من أجل أي $\varepsilon > 0$ يمكن إيجاد عدد مثل δ بحيث إذا كان x يقع ضمن المجال $(a-\delta, a+\delta)$ فإن قيمة الدالة تقع ضمن مجال $(L-\varepsilon, L+\varepsilon)$

لاحظ وفقاً لذلك التعريف أن تحول الدالة x يسعى فقط لـ a دون أن يساويها. يجب النظر عند حساب نهاية أي دالة إلى نقاط انقطاعها (عدم استمرارها) (حيث عند هذه نقاط تكون دالة غير موجودة أو غير معرفة)

مثال: أوجد: $\lim_{x \to 3} \frac{x^2-9}{x-3}$

الحل: بتعويض $x=3$ في المقدار $\frac{x^2-9}{x-3}$ فسوف نصل إلى حالة عدم تعين $\frac{0}{0}$

(انظر حالات عدم التعيين في قسم القوى والجذور من جزء الجبر). لذلك سوف نحلها بطريقة مختلفة.

$$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{(x-3)} = (x+3)$$

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هنا اختصار الكسر صحيح لأن $x \neq 3$ وإنما فقط نعوض 3 والآن نأخذ نهاية:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} (x + 3) = 6$$

لأن إذا كانت x تسعى لـ 3 فإن $x + 3$ تسعى لـ 6.

بعض نهايات شهيرة:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad ; \quad \lim_{x \to 0} \frac{\tan x}{x} = 1 \quad ; \quad \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

قيم لامتناهية في الصغر وقيمة لا نهاية. إذا كان متغيرها نهايته تساوي 0 عندها ندعو هذا المتغير بلامتناهي في الصغر.

مثال: الدالة $y = \sqrt{x+5} - 3$ لامتناهية في الصغر إذا كانت x تسعى إلى 4 لأن

$$\lim_{x \to 4} \sqrt{x + 5} - 3 = 0$$

إذا كانت القيمة المطلقة لمتغيرها يزداد بشكل غير محدود عندها هذا المتغير يدعى بالقيمة لا نهاية (لامتناهي في الكبر).

مثال: الدالة $\frac{x+1}{x-3}$ قيمة لا نهاية إذا كانت x تسعى إلى 3.

القيمة الغير محدودة نهايتها ليست محدودة لكن لها ما يدعى بالنهاية الغير المحدودة وهذه الحقيقة نكتب كالآتي:

$$\lim_{x \to 3} \frac{x+1}{x-3} = \infty$$

الرمز ∞ (لا نهاية) لا تعني عدد ما فقط تعني أن الكسر يزداد بشكل غير محدود إذا كانت x تسعى إلى 3.

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تجدر الإشارة إلى أن الكسر يمكن أن يكون موجب عندما $x>3$ و سالب عندما $x<3$. إذا كانت قيمة لامتناهية يمكن أن تكون موجبة لأي قيم لـ x فإننا سنجلها على سبيل المثال عند $x \to 0$ الدالة $y = x^{-2}$ هي قيمة لامتناهية لكنها موجبة عندما $x>0$ و عندما $x<0$ ويعبر عنها:

$$\lim_{x \to 0} (x^{-2}) = +\infty$$

بالمقابل الدالة $y = -x^{-2}$ دائماً سالبة وعنه

$$\lim_{x \to 0} (-x^{-2}) = -\infty$$

وفقاً لذلك نتيجة في مثالنا السابق يمكن كتابته بالشكل:

$$\lim_{x \to 3} \frac{x+1}{x-3} = \pm\infty$$

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Pronunciation of mathematical expressions

The pronunciations of the most common mathematical expressions are given in the list below. In general, the shortest versions are preferred (unless greater precision is necessary).

1. Logic

$\exists$	there exists
$\forall$	for all
$p \Rightarrow q$	p implies q / if p, then q
$p \Leftrightarrow q$	p if and only if q /p is equivalent to q / p and q are equivalent

2. Sets

$x \in A$	x belongs to A / x is an element (or a member) of A
$x \notin A$	x does not belong to A / x is not an element (or a member) of A
$A \subset B$	A is contained in B / A is a subset of B
$A \supset B$	A contains B / B is a subset of A
$A \cap B$	A cap B / A meet B / A intersection B
$A \cup B$	A cup B / A join B / A union B
$A \setminus B$	A minus B / the difference between A and B
$A \times B$	A cross B / the cartesian product of A and B

3. Real numbers

$x+1$	x plus one
$x-1$	x minus one
$x \pm 1$	x plus or minus one
xy	xy / x multiplied by y
$(x-y)(x+y)$	x minus y, x plus y
$\frac{x}{y}$	x over y
$=$	the equals sign
$x = 5$	x equals 5 / x is equal to 5
$x \neq 5$	x (is) not equal to 5

Good luck :)

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$x \equiv y$	x is equivalent to (or identical with) y
$x \not\equiv y$	x is not equivalent to (or identical with) y
$x > y$	x is greater than y
$x \geq y$	x is greater than or equal to y
$x < y$	x is less than y
$x \leq y$	x is less than or equal to y
$0 < x < 1$	zero is less than x is less than 1
$0 \leq x \leq 1$	zero is less than or equal to x is less than or equal to 1
$\lvert x \rvert$	mod x / modulus x
x^2	x squared / x (raised) to the power 2
x^3	x cubed
x^4	x to the fourth / x to the power four
x^n	x to the nth / x to the power n
x^{-n}	x to the (power) minus n
$\sqrt{x}$	(square) root x / the square root of x
$\sqrt[3]{x}$	cube root (of) x
$\sqrt[4]{x}$	fourth root (of) x
$\sqrt[n]{x}$	nth root (of) x
$(x+y)^2$	x plus y all squared
$\left(\frac{x}{y}\right)^2$	x over y all squared
$n!$	n factorial
$\hat{x}$	x hat
$\bar{x}$	x bar
$\tilde{x}$	x tilde
x_i	xi / x subscript i / x suffix i / x sub i
$\sum_{i=1}^{n} a_i$	the sum from i equals one to n a_i / the sum as i runs from 1 to n of the a_i

4. Linear algebra

$\lVert x \rVert$	the norm (or modulus) of x
$\overrightarrow{OA}$	OA / vector OA
$\overline{OA}$	OA / the length of the segment OA
A^T	A transpose / the transpose of A
A^{-1}	A inverse / the inverse of A

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5. Functions

$f(x)$	fx / f of x / the function f of x
$f : S \to T$	a function f from S to T
$x \mapsto y$	x maps to y / x is sent (or mapped) to y
$f'(x)$	f prime x / f dash x / the (first) derivative of f with respect to x
$f''(x)$	f double-prime x / f double-dash x / the second derivative of f with respect to x
$f'''(x)$	f triple-prime x / f triple-dash x / the third derivative of f with respect to x
$f^{(4)}(x)$	f four x / the fourth derivative of f with respect to x
$\frac{\partial f}{\partial x_1}$	the partial (derivative) of f with respect to x_1
$\frac{\partial^2 f}{\partial x_1^2}$	the second partial (derivative) of f with respect to x_1
$\int_0^\infty$	the integral from zero to infinity
$\lim_{x \to 0}$	the limit as x approaches zero
$\lim_{x \to +0}$	the limit as x approaches zero from above
$\lim_{x \to -0}$	the limit as x approaches zero from below
$\log_e y$	log y to the base e / log to the base e of y / natural log (of) y
$\ln y$	log y to the base e / log to the base e of y / natural log (of) y

Individual mathematicians often have their own way of pronouncing mathematical expressions and in many cases there is no generally accepted "correct" pronunciation.

Distinctions made in writing are often not made explicit in speech; thus the sounds fx may be interpreted as any of: fx, $f(x)$, f_x, FX, $\overline{FX}$, $\overrightarrow{FX}$. The difference is usually made clear by the context; it is only when confusion may occur, or where he/she wishes to emphasise the point, that the mathematician will use the longer forms: f multiplied by x, the function f of x, f subscript x, line FX, the length of the segment FX, vector FX.

Similarly, a mathematician is unlikely to make any distinction in speech (except sometimes a difference in intonation or length of pauses) between pairs such as the following:

$$x + (y + z) \quad \text{and} \quad (x + y) + z$$
$$\sqrt{ax} + b \quad \text{and} \quad \sqrt{ax + b}$$
$$a^n - 1 \quad \text{and} \quad a^{n-1}$$

The primary reference has been David Hall with Tim Bowyer, Nucleus. English for Science and Technology. Mathematics. Longman 1980. Glen Anderson and Matti Vuorinen have given good comments and supplements.

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