Convexity, concavity and inflexion points of a function

The second derivative. Convex and concave function.

Sufficient condition of concavity (convexity) of a function.

Inflexion point.

The second derivative. If a derivative $f'(x)$ of a function $f(x)$ is differentiable in the point (x_0), then its derivative is called the second derivative of the function $f(x)$ in the point (x_0) and denoted as $f''(x_0)$.

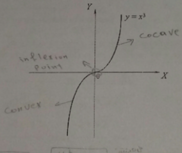
A function $f(x)$ is called *convex* in an interval (a, b), if a graph of the function $f(x)$ is placed in this interval *lower* than a tangent line, going through any point $(x_0, f(x_0))$, $x_0 \in (a, b)$.

A function $f(x)$ is called *concave* in an interval (a, b), if a graph of the function $f(x)$ is placed in this interval *higher* than a tangent line, going through any point $(x_0, f(x_0))$, $x_0 \in (a, b)$.

Sufficient condition of concavity (convexity) of a function. Let a function $f(x)$ be twice differentiable (i.e. it has the *second* derivative) in an interval (a, b), then: if $f''(x) > 0$ for any $x \in (a, b)$, then the function $f(x)$ is *concave* in the interval (a, b); if $f''(x) < 0$ for any $x \in (a, b)$, then the function $f(x)$ is *convex* in the interval (a, b).

If a function changes a convexity to a concavity or vice versa at passage through some point, then this point is called ***an inflexion point*** an inflexion point. Hence it follows, that if the second derivative f'' exists in an inflexion point x_0, then $f''(x_0) = 0$.

E x a m p l e. Consider a graph of the function $y = x^3$:



This function is concave at $x > 0$ and convex at $x < 0$. In fact, $y'' = 6x$, but $6x > 0$ at $x > 0$ and $6x < 0$ at $x < 0$, hence, $y'' > 0$ at $x > 0$ and $y'' < 0$ at $x < 0$, hence it follows, that the function $y = x^3$ is concave at $x > 0$ and convex at $x < 0$. Then the point $x = 0$ is the inflexion point of the function $y = x^3$.

Primitive. Indefinite integral

Primitive. Finding of primitive; infinite set of solutions.
Indefinite integral. Constant of integration.

Primitive. A continuous function $F(x)$ is called a *primitive* for a function $f(x)$ on a segment X, if for each $x \in X$

$$F'(x) = f(x).$$

E x a m p l e . The function $F(x) = x^3$ is a primitive for the function $f(x) = 3x^2$ on the interval $(-\infty, +\infty)$, because

$$F'(x) = (x^3)' = 3x^2 = f(x)$$

for all $x \in (-\infty, +\infty)$.
It is easy to check, that the function $x^3 + 13$ has the same derivative $3x^2$, so it is also a primitive for the function $3x^2$ for all $x \in (-\infty, +\infty)$.

It is clear, that instead of 13 we can use any constant. Thus, *the problem of finding a primitive has an infinite set of solutions.* This fact is reflected in the definition of an *indefinite integral:*

Indefinite integral of a function $f(x)$ on a segment X is *a set of all its primitives*. This is written as :

$$\int f(x)\,dx = F(x) + C,$$

where C – any constant, called a *constant of integration*.

Basic properties of indefinite integral

If a function $f(x)$ has a primitive on a segment X, and k – a number, then

$$\int k\,f(x)\,dx = k \int f(x)\,dx.$$

If functions $f(x)$ and $g(x)$ have primitives on a segment X, then

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

If a function $f(x)$ has a primitive on a segment X, then for interior points of this segment:

$$\frac{d}{dx} \int f(x)\,dx = f(x).$$

If $f(x)$ is a continuous function on a segment X and differentiable in interior points of this segment, then:

$$\int df(x) = f(x) + C.$$

Definite integral. Newton – Leibniz formula

Curvilinear trapezoid. Area of a curvilinear trapezoid. Definite integral. Limits of integration. Integrand. Newton-Leibniz formula.

Consider a continuous function $y = f(x)$, given on a segment $[a, b]$ and saving its sign on this segment (Fig.8). The figure, bounded by a graph of this function, a segment $[a, b]$ and straight lines $x = a$ and $x = b$, is called a *curvilinear trapezoid*. To calculate areas of curvilinear trapezoids the following theorem is used:

If f – a continuous, non-negative function on a segment $[a, b]$, and F – its primitive on this segment, then an area S of the corresponding curvilinear trapezoid is equal to an increment of the primitive on a segment $[a, b]$, i.e.

$$S = F(b) - F(a)$$

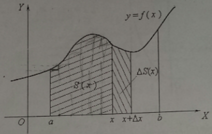


Fig. 8

Consider a function $S(x)$, given on a segment $[a, b]$. If $a < x \leq b$, then $S(x)$ is an area of the part of the curvilinear trapezoid, which is placed on the left of a vertical straight line, going through the point $(x, 0)$. Note, that if $x = a$, then $S(a) = 0$ and $S(b) = S$ (S – area of the curvilinear trapezoid). It is possible to prove, that

$$\lim_{\Delta x \to 0} \frac{S(x + \Delta x) - S(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta S(x)}{\Delta x} = f(x),$$

or shortly:

$$S'(x) = f(x);$$

i.e. $S(x)$ is a primitive for $f(x)$. Hence, according to the basic property of primitives, for all $x \in [a, b]$ we have:

$$S(x) = F(x) + C,$$

where C – some constant, F – one of the primitives for a function f.

To find C we substitute $x = a$:

$$F(a) + C = S(a) = 0.$$

hence, $C=-F(a)$ and $S(x)=F(x)-F(a)$. As an area of the curvilinear trapezoid is equal to $S(b)$, substituting $x=b$, we'll receive:

$$\boxed{S=S(b)=F(b)-F(a).}$$

E x a m p l e. Find an area of a figure, bounded by the curve $y=x^2$ and lines $y=0, x=1, x=2$ (Fig.9).

S o l u t i o n.

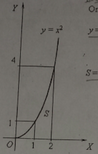


Fig. 9

One of primitives for the function $y=x^2$ is $F(x)=x^3/3$. Then

$$S=F(2)-F(1)=\frac{2^3}{3}-\frac{1^3}{3}=\frac{7}{3}.$$

Definite integral. Consider another way to calculate an area of a curvilinear trapezoid. Divide a segment $[a, b]$ into n segments of an equal length by points:

$$x_0=a<x_1<x_2<x_3<\ldots<x_{n-1}<x_n=b$$

and let $\Delta x=(b-a)/n=x_k-x_{k-1}$, where $k=1, 2, \ldots, n-1, n$. In each of segments $[x_{k-1}, x_k]$ as on a base we'll build a rectangle of height $f(x_{k-1})$. An area of this rectangle is equal to:

$$f(x_{k-1})\Delta x=\frac{b-a}{n}f(x_{k-1});$$

and a sum of areas of these rectangles (Fig.10) is equal to:

$$S_n=\frac{b-a}{n}[f(x_0)+f(x_1)+\ldots+f(x_{n-1})].$$

In view of continuity of a function $f(x)$ a union of the built rectangles at great n (i.e. at small Δx) "almost coincides" with our curvilinear trapezoid. Therefore, $S_n \approx S$ at great values of n. It means, that $S_n \to S$ at $n\to\infty$. This limit is called *an integral of a function $f(x)$ from a to b* or *a definite integral*:

$\int_a^b f(x)\,dx$, i.e. $S_n \to \int_a^b f(x)\,dx$ at $n \to \infty$.

Numbers a and b are called *limits of integration*, $f(x)\,dx$ – an *integrand*. So, if $f(x) \geq 0$ on a segment $[a, b]$, then an area S of the corresponding curvilinear trapezoid is represented by the formula:

$$S = \int_a^b f(x)\,dx.$$

Newton – Leibniz formula. Comparing the two formulas of a curvilinear trapezoid area, we make the conclusion: if $F(x)$ is primitive for the function $f(x)$ on a segment $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This is the famous *Newton – Leibniz formula.* It is valid for any function $f(x)$, which is continuous on a segment $[a, b]$.

E x a m p l e. Calculate the integral: $\int_0^{\pi} \sin x\,dx$.

S o l u t i o n. Using the table of integrals for some elementary functions (see above), we'll receive:

$$\int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi} = -\cos\pi - (-\cos 0) = 2.$$

رياضيات ٢ اللغة الانكليزية جبر العام الثاني

ص٣

MATRICES

Matrix مصفوفة

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Chapter I: Introduction of Matrices

1.1 Definition 1:

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within two bracket is called a matrix. We shall denote matrices by capital letters as A,B, C etc.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m \times n}$$

A is a matrix of order $m \times n$, i^{th} row j^{th} column element of the matrix denoted by a_{ij}

Remark: A matrix is not just a collection of elements but every element has assigned a definite position in a particular row and column.

1.2 Special Types of Matrices:

1. **Square matrix:**

   A matrix in which numbers of rows are equal to number of columns is called a square matrix.

2. **Diagonal matrix:**

   A square matrix $A = (a_{ij})_{n \times n}$ is called a diagonal matrix if each of its non-diagonal element is zero.

   That is $a_{ij} = 0$ if $i \neq j$ and at least one element $a_{ii} \neq 0$.

3. **Identity Matrix**

   A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n.

   That is $a_{ij} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$

4. **Upper Triangular matrix:**

   A square matrix said to be a Upper triangular matrix if $a_{ij} = 0$ if $i > j$.

5. **Lower Triangular Matrix:**

A square matrix said to be a Lower triangular matrix if $a_{ij} = 0$ *if* $i < j$.

6. **Symmetric Matrix:**

A square matrix $A = (a_{ij})_{n \times n}$ said to be a symmetric if $a_{ij} = a_{ji}$ for all i and j.

7. **Skew- Symmetric Matrix:**

A square matrix $A = (a_{ij})_{n \times n}$ said to be a skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j.

8. **Zero Matrix:**

A matrix whose all elements are zero is called as Zero Matrix and order $m \times n$ Zero matrix denoted by $0_{n \times m}$.

9. **Row Vector**

A matrix consists a single row is called as a row vector or row matrix.

10. **Column Vector**

A matrix consists a single column is called a column vector or column matrix.

Chapter 2: Matrix Algebra

2.1. Equality of two matrices:

Two matrices A and B are said to be equal if

(i) They are in same order.

(ii) Their corresponding elements are equal.

That is if $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ then $a_{ij} = b_{ij}$ for all i and j.

2.2. Scalar multiple of a matrix

Let k be a scalar then scalar product of matrix $A = (a_{ij})_{m \times n}$ given denoted by kA and given by $kA = (ka_{ij})_{m \times n}$

2.3. Addition of two matrices:

Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ are two matrices with same order then sum of the two matrices are given by

$$A + B = (a_{ij})_{m \times n} + (b_{ij})_{m \times n} = (a_{ij} + b_{ij})_{m \times n}$$

2.4. Multiplication of two matrices:

Two matrices A and B are said to be confirmable for product AB if number of columns in A equals to the number of rows in matrix B. Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{n \times r}$ be two matrices the product matrix $C = AB$, is matrix of order $m \times r$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots\dots\dots + a_{in}b_{nj}$$

2.5. Integral power of Matrices:

Let A be a square matrix of order n, and m be positive integer then we define

$$A^m = A \times A \times A \dots\dots \times A \qquad \text{(m times multiplication)}$$

2.6. Properties of the Matrices

Let A, B and C are three matrices and λ and μ are scalars then

(i) $A + (B + C) = (A + B) + C$

(ii) $\lambda(A + B) = \lambda A + \lambda B$

(iii) $\lambda(\mu A) = (\lambda\mu)A$

(iv) $(\lambda A)B = \lambda(AB)$

(v) $A(BC) = (AB)C$

(vi) $A(B + C) = AB + AC$

2.7. Transpose:

The transpose of matrix $A = (a_{ij})_{m \times n}$, written A^t (A' or A^T) is the matrix obtained by writing the rows of A in order as columns.

That is $A^t = (a_{ji})_{n \times m}$.

Properties of Transpose:

(i) $(A + B)^t = (A^t + B^t)$

(ii) $(A^t)^t = A$

(iii) $(kA)^t = k A^t$ for scalar k.

(iv) $(AB)^t = B^t A^t$

2.8 A square matrix A is said to be symmetric if $A = A^t$.

2.9 A square matrix A is said to be skew- symmetric if $A = -A^t$

(i) AA^t and A^tA are both symmetric.

(ii) $A + A^t$ is a symmetric matrix.

(iii) $A - A^t$ is a skew-symmetric matrix.

(iv) If A is a symmetric matrix and m is any positive integer then A^m is also symmetric.

(v) If A is skew symmetric matrix then odd integral powers of A is skew symmetric, while positive even integral powers of A is symmetric.

If A and B are symmetric matrices then

(vi) $(AB + BA)$ is symmetric.

(vii) $(AB - BA)$ is skew-symmetric.

Chapter 3: Determinant, Minor and Adjoint Matrices

Definition 3.1:

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n , then the number $|A|$ called determinant of the matrix A.

(i) Determinant of 2×2 matrix

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

(ii) Determinant of 3×3 matrix

Let $B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then $|B| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$$|B| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

3.1 Properties of the Determinant:

a. The determinant of a matrix A and its transpose A^t are equal.

$$|A| = |A^t|$$

b. Let A be a square matrix

(i) If A has (column) of zeros then $|A| = 0$.

(ii) If A has two identical rows (or columns) then $|A| = 0$.

c. If A is triangular matrix then $|A|$ is product of the diagonal elements.

d. If A is a square matrix of order n and k is a scalar then $|kA| = k^n|A|$

3.2 Singular Matrix

If A is square matrix of order n, the A is called singular matrix when $|A| = 0$ and non-singular otherwise.

3.3. Minor and Cofactors:

Let $A = (a_{ij})_{n \times n}$ is a square matrix. Then M_{ij} denote a sub matrix of A with order $(n-1) \times (n-1)$ obtained by deleting its i^{th} row and j^{th} column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A.

The cofactor of a_{ij} denoted by A_{ij} and is equal to $(-1)^{i+j}|M_{ij}|$.

3.4. Adjoin Matrix:

The transpose of the matrix of cofactors of the element a_{ij} of A denoted by $adj\ A$ is called adjoin of matrix A.

Theorem 3.1:

For any square matrix A,

$$A\,(adj\ A) = (adj\ A)A = |A|\ I$$ where I is the identity matrix of same order.

Theorem 3.2: If A is a non-singular matrix of order n, then $|adj\ A| = |A|^{n-1}$.

Theorem 3.3: If A and B are two square matrices of order n then

$$adj(AB) = (adj\ B)(adj\ A)$$

$$= |AB|I \quad \text{.............. (ii)}$$

Some results of adjoint

(i) For any square matrix A. $(adj\ A)^t = adj\ A^t$

(ii) The adjoint of an identity matrix is the identity matrix.

(iii) The adjoint of a symmetric matrix is a symmetric matrix.

Chapter 4: Inverse of a Matrix and Elementary Row Operations

4.1 Inverse of a Matrix

Definition 4.1:

If A and B are two matrices such that $AB = BA = I$, then each is said to be inverse of the other. The inverse of A is denoted by A^{-1}.

Theorem 4.1: (Existence of the Inverse)

The necessary and sufficient condition for a square matrix A to have an inverse is that $|A| \neq 0$ (That is A is non singular).

Theorem 4.2: (Uniqueness of the Inverse)

Inverse of a matrix if it exists is unique.

Theorem 4.3: (Reversal law of the inverse of product)

If A and B are two non-singular matrices of order n, then (AB) is also non singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Corollary4.1: If $A_1A_2 \dots\dots A_m$ are non singular matrices of order n, then $(A_1A_2 \dots\dots A_m)^{-1} = A_m^{-1}A_{m-1}^{-1} \dots\dots A_1^{-1}$.

Theorem 4.4: If A is a non-singular matrix of order n then $(A^t)^{-1} = (A^{-1})^t$.

Theorem 4.5: If A is a non-singular matrix, k is non zero scalar, then $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Theorem 4.6: If A is a non-singular matrix then

$$|A^{-1}| = \frac{1}{|A|}.$$

4.2 Elementary Transformations:

Some operations on matrices called as elementary transformations. There are six types of elementary transformations, three of then are row transformations and other three of them are column transformations. There are as follows

(i) Interchange of any two rows or columns.

(ii) Multiplication of the elements of any row (or column) by a non zero number k.

(iii) Multiplication to elements of any row or column by a scalar k and addition of it to the corresponding elements of any other row or column.

We adopt the following notations for above transformations

(i) Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$. $V_i \leftrightarrow V_j$

(ii) Multiplication by k to all elements in the i^{th} row $R_i \to kR_i$. $V_i \leftrightarrow kV_i$

(iii) Multiplication to elements of j^{th} row by k and adding them to the corresponding elements of i^{th} row is denoted by $R_i \to R_i + kR_j$.

4.2.1 Equivalent Matrix:

$V_i \to V_i + kV_j$

A matrix B is said to be equivalent to a matrix A if B can be obtained from A, by for forming finitely many successive elementary transformations on a matrix A.

Denoted by A~ B.

4.3 Rank of a Matrix:

Definition:

A positive integer 'r' is said to be the rank of a non- zero matrix A if

(i) There exists at least one non-zero minor of order r of A and

(ii) Every minor of order greater than r of A is zero.

minor ?

The rank of a matrix A is denoted by $\rho(A)$.

Rank رتبة

order رتبة

PLUS

4.4 Echelon Matrices:

Definition 4.3:

A matrix $A = (a_{ij})$ is said to be echelon form (echelon matrix) if the number of zeros preceding the first non zero entry of a row increasing by row until zero rows remain.

In particular, an echelon matrix is called a row reduced echelon matrix if the distinguished elements are

(i) The only non- zero elements in their respective columns.

(ii) Each equal to 1.

Remark: The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

Chapter 5: Solution of System of Linear Equation by Matrix Method

5.1 Solution of the linear system $AX=B$

We now study how to find the solution of system of m linear equations in n unknowns.

Consider the system of equations in unknowns $x_1, x_2, \dots\dots, x_n$ as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots\dots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots a_{nn}x_n = b_n \end{cases}$$

is called system of linear equations with n unknowns $x_1, x_2, \dots\dots, x_n$. If the constants $b_1, b_2, \dots\dots b_n$ are all zero then the system is said to be homogeneous type.

The above system can be put in the matrix form as

$AX = B$

Where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ $B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$

The matrix $A = (a_{ij})_{n\times n}$ is called coefficient matrix, the matrix X is called matrix of unknowns and B is called as matrix of constants, matrices X and B are of order $n \times 1$.

Definition 5.1: (consistent)

A set of values of $x_1, x_2, \dots\dots\dots, x_n$ which satisfy all these equations simultaneously is called the solution of the system. If the system has at least one solution then the equations are said to be consistent otherwise they are said to be inconsistent.

نظرية

Theorem 5.2:

A system of m equations in n unknowns represented by the matrix equation $AX=B$ is consistent if and only if $\rho(A) = \rho(A,B)$. That is the rank of matrix A is equal to rank of augment matrix (A,B)

نظرية

Theorem 5.3:

If A be an non-singular matrix, X be an $n \times 1$ matrix and B be an $n \times 1$ matrix then the system of equations $AX=B$ has a unique solution.

حل وحيد

(1)

$$AX = B$$

- محلولة: Consistent if $\rho(A) = \rho(A,B) = r$
  - حل وحيد: Unique solution if r=n
  - حلول عدد غير منتهي: Infinite solution if r<n
- Inconsistent if $\rho(A) \neq \rho(A,B)$

(2)

محلولة بجميع الطرق

$$AX = 0 \text{ (All wayes consistant)}$$

- If $\rho(A) = r = n$ Trivial solution
- If $\rho(A) = r < n$ Infinite solutions

Therefore every system of linear equations solutions under one of the following:

(i) There is no solution

(ii) There is a unique solution

(iii) There are more than one solution

Methods of solving system of linear Equations:

5.1 Method of inversesion:

Consider the matrix equation

Consider the matrix equation

$AX = B$ when $|A| \neq 0$ then the system has a unique solution.

Pre multiplying by A^{-1}, we have

$$A^{-1}(AX) = A^{-1}B$$

$$X = A^{-1}B$$

Thus $AX = B$, has only one solution if $|A| \neq 0$ and is given by $\boxed{X = A^{-1}B}$.

5.2 Using Elementary row operations: (Gaussian Elimination)

Suppose the coefficient matrix is of the type $m \times n$. That is we have m equations in n unknowns Write matrix $[A, B]$ and reduce it to Echelon augmented form by applying elementary row transformations only.

5.3 Cramer's Rule

When $|A| \neq 0$ then the system has a unique solution. And the solution can determine with out computing A^{-1} or using row reduction.

Let $D_i = \det A_i \qquad i = 1,2,3 \ldots\ldots n$

Let A be an matrix and suppose that $|A| \neq 0$ then the unique solution to the system $AX = B$ is given by

$$x_i = \frac{D_i}{|A|} \qquad i = 1, 2, 3, \ldots\ldots n$$

Chapter 6: Eigenvalues and Eigenvectors and Diagonalization:

If A is a square matrix of order n and X is a vector in R^n, (X considered as $n \times 1$ column matrix), we are going to study the properties of non-zero X, where AX are scalar multiples of one another. Such vectors arise naturally in the study of vibrations, electrical systems, genetics, chemical reactions, quantum mechanics, economics and geometry.

Definition 6.1:

If A is a square matrix of order n , then a non-zero vector X in R^n is called eigenvector of A if $AX = \lambda X$ for some scalar λ. The scalar λ is called an eigenvalue of A, and X is said to be an eigenvector of A corresponding to λ.

Remark: Eigen values are also called proper values or characteristic values.

Theorem 6.1: If A is a square matrix of order n and λ is a real number, then λ is an eigenvalue of A if and only if $|\lambda I - A| = 0$.

Theorem 6.2:

If A is an $n \times n$ matrix and λ is a real number, then the following are equivalent:

(i) λ is an eigenvalue of A.
(ii) The system of equations $(\lambda I - A)X = 0$ has non-trivial solutions.
(iii) There is a non-zero vector X in R^n such that $AX = \lambda I$.
(iv) Is a solution of the characteristic equation $|(A - \lambda I)| = 0$.

Definition 6.2:

Let A be an $n \times n$ matrix and λ be the eigen value of A. The set of all vectors X in R^n which satisfy the identity $AX = \lambda X$ is called the eigen space of a corresponding to λ. This is denoted by $E(\lambda)$.

تذكرة

Remark:

الشعاع الذاتي المقابل للقيمة الذاتية هي أشعة غير صفرية من التي تحقق

The eigenvectors of A corresponding to an eigen value λ are the non-zero vectors of X that satisfy $AX = \lambda I$, Equivalently the eigen vectors corresponding to λ are the non zero in the solution space of $(\lambda I - A)X = 0$. Therefore, the eigen space is the set of all non-zero X that satisfy $(A - \lambda I)X = 0$ with trivial solution in addition.

الفضاء الذاتي هو مجموعة كل الأشعة غير صفرية التي تحقق ... مع الحل البسيط "الناقصة" أي أجمع

التقطير

Diagonalization:

تعريف

Definition 7.1

Two matrices A and B are said to be similar there exists an invertible matrix C such that

$$B = C^{-1}AC$$

نقول عن مصفوفتين A و B أنهما متشابهتين إذا وجدت مصفوفة مقلوبة C بحيث $B = C^{-1}AC$

مبرهنة: إذا كانت A و B مصفوفتين متشابهتين

نظرية

Theorem 7.1

If A and B are similar matrices, then A and B have the same characteristic equation and have the same eigen values.

عندئذ A و B لديهما المعادلة المميزة نفسها ونفس القيم الذاتية

تعريف

Definition 7.2: A square matrix A is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix, the matrix P is said to diagonalizable A.

Theorem 7.1: If A is a square matrix of order n, then the following are equivalent.

متكافئة الشروط التالية

(i) A is diagonizible.

(ii) A has n linearly independent eigenvectors.

لديها n شعاع ذاتي مستقل خطياً

تعريف:

نقول عن المصفوفة المربعة أنها قابلة للتقطير إذا وجدت مصفوفة مقلوبة P بحيث تحقق $(P^{-1}AP)$ مصفوفة قطرية أي:

$$P = P^{-1}AP$$

GOOD LUCK

Reem math 4 ever

PLUS

Appendix A

Sets and Relations

SETS, ELEMENTS

Any well defined list or collection of objects is called a *set*; the objects comprising the set are called its *elements* or *members*. We write

$$p \in A \quad \text{if } p \text{ is an element in the set } A$$

If every element of A also belongs to a set B, i.e. if $x \in A$ implies $x \in B$, then A is called a *subset* of B or is said to be *contained* in B; this is denoted by

$$A \subset B \quad \text{or} \quad B \supset A$$

Two sets are *equal* if they both contain the same elements; that is,

$$A = B \quad \text{if and only if} \quad A \subset B \text{ and } B \subset A$$

The negations of $p \in A$, $A \subset B$ and $A = B$ are written $p \notin A$, $A \not\subset B$ and $A \neq B$ respectively.

We specify a particular set by either listing its elements or by stating properties which characterize the elements in the set. For example,

$$A = \{1, 3, 5, 7, 9\}$$

means A is the set consisting of the numbers 1, 3, 5, 7 and 9; and

$$B = \{x : x \text{ is a prime number}, x < 15\}$$

means that B is the set of prime numbers less than 15. We also use special symbols to denote sets which occur very often in the text. Unless otherwise specified:

- $\mathbf{N}$ = the set of positive integers: $1, 2, 3, \ldots$;
- $\mathbf{Z}$ = the set of integers: $\ldots, -2, -1, 0, 1, 2, \ldots$;
- $\mathbf{Q}$ = the set of rational numbers;
- $\mathbf{R}$ = the set of real numbers;
- $\mathbf{C}$ = the set of complex numbers.

We also use $\emptyset$ to denote the *empty* or *null* set, i.e. the set which contains no elements; this set is assumed to be a subset of every other set.

Frequently the members of a set are sets themselves. For example, each line in a set of lines is a set of points. To help clarify these situations, we use the words *class*, *collection* and *family* synonymously with set. The words *subclass*, *subcollection* and *subfamily* have meanings analogous to subset.

Example A.1: The sets A and B above can also be written as

$$A = \{x \in \mathbf{N} : x \text{ is odd}, x < 10\} \quad \text{and} \quad B = \{2, 3, 5, 7, 11, 13\}$$

Observe that $9 \in A$ but $9 \notin B$, and $11 \in B$ but $11 \notin A$; whereas $3 \in A$ and $3 \in B$, and $6 \notin A$ and $6 \notin B$.

Example A.2: The sets of numbers are related as follows: $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$.

Example A.3: Let $C = \{x : x^2 = 4, x \text{ is odd}\}$. Then $C = \emptyset$, that is, C is the empty set.

Example A.4: The members of the class $\{\{2,3\}, \{2\}, \{5,6\}\}$ are the sets $\{2,3\}$, $\{2\}$ and $\{5,6\}$.

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The following theorem applies.

Theorem A.1: Let A, B and C be any sets. Then: (i) $A \subset A$; (ii) if $A \subset B$ and $B \subset A$, then $A = B$; and (iii) if $A \subset B$ and $B \subset C$, then $A \subset C$.

We emphasize that $A \subset B$ does not exclude the possibility that $A = B$. However, if $A \subset B$ but $A \neq B$, then we say that A is a *proper subset* of B. (Some authors use the symbol $\subseteq$ for a subset and the symbol $\subset$ only for a proper subset.)

When we speak of an *indexed set* $\{a_i : i \in I\}$, or simply $\{a_i\}$, we mean that there is a mapping ϕ from the set I to a set A and that the image $\phi(i)$ of $i \in I$ is denoted a_i. The set I is called the *indexing set* and the elements a_i (the range of ϕ) are said to be *indexed* by I. A set $\{a_1, a_2, \ldots\}$ indexed by the positive integers $\mathbf{N}$ is called a *sequence*. An indexed class of sets $\{A_i : i \in I\}$, or simply $\{A_i\}$, has an analogous meaning except that now the map ϕ assigns to each $i \in I$ a set A_i rather than an element a_i.

SET OPERATIONS

Let A and B be arbitrary sets. The *union* of A and B, written $A \cup B$, is the set of elements belonging to A or to B; and the *intersection* of A and B, written $A \cap B$, is the set of elements belonging to both A and B:

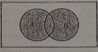
$$A \cup B = \{x : x \in A \text{ or } x \in B\} \quad \text{and} \quad A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If $A \cap B = \emptyset$, that is, if A and B do not have any elements in common, then A and B are said to be *disjoint*.

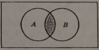
We assume that all our sets are subsets of a fixed *universal set* (denoted here by U). Then the *complement* of A, written A^c, is the set of elements which do not belong to A:

$$A^c = \{x \in U : x \notin A\}$$

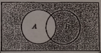
Example A.5: The following diagrams, called Venn diagrams, illustrate the above set operations. Here sets are represented by simple plane areas and U, the universal set, by the area in the entire rectangle.



$A \cup B$ is shaded



$A \cap B$ is shaded



A^c is shaded

لتكن A و B و C أي مجموعات عندئذ:
1) $A \subset A$ الانعكاسية
2) إذا كان $A \subset B$ و $B \subset A$ عندها $A = B$ (تخالفية)
3) إذا كان $A \subset B$ و $B \subset C$ عندها $A \subset C$ (تعدية)

لنأكد أن $A \subset B$ لا نستبعد إمكانية هذا
$A = B$ لكن إذا كان $A \subset B$
لكن $A \neq B$ عندها نقول أن A جزء خاص (حقيقي) من B.
(بعض الكتب تستخدم الرمز ($\subseteq$) للمجموعة الجزئية والرمز ($\subset$) لنقول أن A جزء خاص (حقيقي) من B.)

عندما نتكلم عن مجموعة مرقمة {أو بشكل مختصر $\{a_i\}$}
نعني أنه يوجد تطبيق ϕ من مجموعة I إلى مجموعة A وأن الصورة $\phi(i)$ لـ $i \in I$ نرمز لها a_i. المجموعة I تدعى مجموعة مرقمة والعناصر a_i (المستقر لـ ϕ) نقول عنها مرقمة بـ I، $\{A$
مرقمة بأعداد صحيحة موجبة N تدعى متتالية.

Sets under the above operations satisfy various laws or identities which are listed in the table below. In fact, we state

Theorem A.2: Sets satisfy the laws in Table 1.

LAWS OF THE ALGEBRA OF SETS	
Idempotent Laws	
1a. $A \cup A = A$	1b. $A \cap A = A$
Associative Laws	
2a. $(A \cup B) \cup C = A \cup (B \cup C)$	2b. $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws	
3a. $A \cup B = B \cup A$	3b. $A \cap B = B \cap A$
Distributive Laws	
4a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	4b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Laws	
5a. $A \cup \emptyset = A$	5b. $A \cap U = A$
6a. $A \cup U = U$	6b. $A \cap \emptyset = \emptyset$
Complement Laws	
7a. $A \cup A^c = U$	7b. $A \cap A^c = \emptyset$
8a. $(A^c)^c = A$	8b. $U^c = \emptyset$, $\emptyset^c = U$
De Morgan's Laws	
9a. $(A \cup B)^c = A^c \cap B^c$	9b. $(A \cap B)^c = A^c \cup B^c$

Table 1

Remark: Each of the above laws follows from an analogous logical law. For example,

$$A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \in A\} = B \cap A$$

(Here we use the fact that the composite statement "p and q", written $p \wedge q$, is logically equivalent to the composite statement "q and p", i.e. $q \wedge p$.)

The relationship between set inclusion and the above set operations follows.

Theorem A.3: Each of the following conditions is equivalent to $A \subset B$:

(i) $A \cap B = A$ (iii) $B^c \subset A^c$ (v) $B \cup A^c = U$

(ii) $A \cup B = B$ (iv) $A \cap B^c = \emptyset$

We generalize the above set operations as follows. Let $\{A_i : i \in I\}$ be any family of sets. Then the *union* of the A_i, written $\bigcup_{i \in I} A_i$ (or simply $\bigcup_i A_i$), is the set of elements each belonging to at least one of the A_i; and the *intersection* of the A_i, written $\bigcap_{i \in I} A_i$ or simply $\bigcap_i A_i$, is the set of elements each belonging to every A_i.

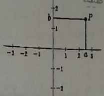
PRODUCT SETS

Let A and B be two sets. The *product set* of A and B, denoted by $A \times B$, consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

The product of a set with itself, say $A \times A$, is denoted by A^2.

Example A.6: The reader is familiar with the cartesian plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ as shown below. Here each point P represents an ordered pair (a, b) of real numbers, and vice versa.



Example A.7: Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then

$$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$$

Remark: The ordered pair (a, b) is defined rigorously by $(a, b) \equiv \{\{a\}, \{a, b\}\}$. From this definition, the "order" property may be proven; that is, $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

The concept of product set is extended to any finite number of sets in a natural way. The *product set* of the sets $A_1, \ldots, A_m$, written $A_1 \times A_2 \times \cdots \times A_m$, is the set consisting of all m-tuples $(a_1, a_2, \ldots, a_m)$ where $a_i \in A_i$ for each i.

RELATIONS

A *binary relation* or simply *relation* R from a set A to a set B assigns to each ordered pair $(a, b) \in A \times B$ exactly one of the following statements:

(i) "a is related to b", written $a\,R\,b$,

(ii) "a is not related to b", written $a \not R\, b$.

A relation from a set A to the same set A is called a *relation in* A.

Example A.8: Set inclusion is a relation in any class of sets. For, given any pair of sets A and B, either $A \subset B$ or $A \not\subset B$.

Observe that any relation R from A to B uniquely defines a subset $\hat{R}$ of $A \times B$ as follows:

$$\hat{R} = \{(a, b) : a\,R\,b\}$$

Conversely, any subset $\hat{R}$ of $A \times B$ defines a relation from A to B as follows:

$$a\,R\,b \text{ if and only if } (a, b) \in \hat{R}$$

In view of the above correspondence between relations from A to B and subsets of $A \times B$, we redefine a relation as follows:

Definition: A relation R from A to B is a subset of $A \times B$.

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