المحاضرة 14 — الخميس 19/5/2016

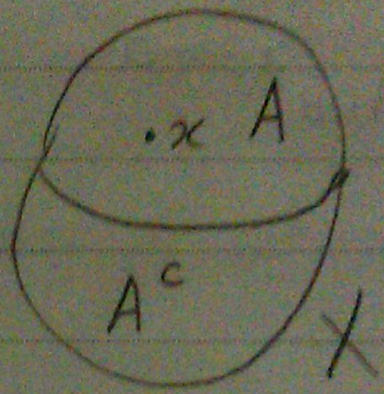
تكامل لوبيغ (إيجاد قوى تركيب خطي لتوابع درجية)

- التابع المميز (الدرجي)
- التابع البسيط
- تكامل لوبيغ لتابع بسيط

$X \neq \emptyset$

$A \subseteq X \qquad \chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in A^c \end{cases}$

$\mathcal{A} = \{X, \emptyset, A, A^c\}$



$\chi_X(x) = 1 \quad , \quad \chi_\emptyset(x) = 0$

① $\chi_{A\cap B}(x) = \chi_A(x) \cdot \chi_B(x)$

② $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) \quad ; \quad A \cap B = \emptyset$

الإثبات:

1) $x \in A \cap B \Rightarrow \chi_{A\cap B}(x) = 1$

$\left.\begin{array}{l} x \in A \Rightarrow \chi_A(x) = 1 \\ x \in B \Rightarrow \chi_B(x) = 1 \end{array}\right\} \quad 1 \cdot 1 = 1$

$x \notin A \cap B \Rightarrow \chi_{A\cap B}(x) = 0$

$\left.\begin{array}{l} x \notin A \wedge x \notin B \\ x \in A \wedge x \notin B \\ x \notin A \wedge x \in B \end{array}\right\} \begin{array}{lll} \chi_A(x) = 0 \,,\ \chi_B(x) = 0 & \Rightarrow & 0 \cdot 0 = 0 \\ \chi_A(x) = 1 \,,\ \chi_B(x) = 0 & \Rightarrow & 0 \cdot 1 = 0 \\ \chi_A(x) = 0 \,,\ \chi_B(x) = 1 & \Rightarrow & 0 \cdot 1 = 0 \end{array}$

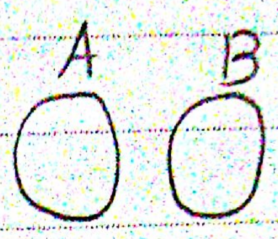
Scanned by CamScanner

2) $A \cap B = \emptyset$

* $x \in A \cup B$ $\qquad \chi_{A \cup B}(x) = 1$

$x \in A \wedge x \notin B \Rightarrow 1 + 0 = 1$

$x \notin A \wedge x \in B \Rightarrow 0 + 1 = 1$

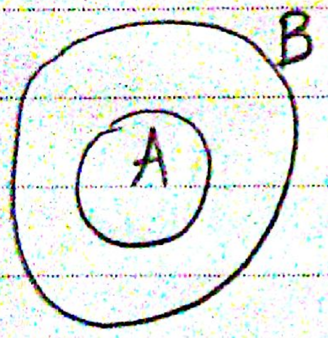


* $x \notin A \cup B$

$\Rightarrow \chi_{A \cup B}(x) = 0 \qquad : x \notin A \wedge x \notin B$

$$\Rightarrow \left\{ \begin{array}{l} \chi_A(x) = 0 \\ \chi_B(x) = 0 \end{array} \right\} \Rightarrow 0 + 0 = 0$$

③ $A \subseteq B \qquad \chi_A(x) \leq \chi_B(x)$

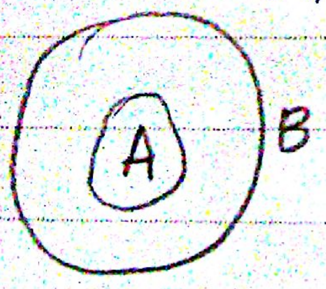


$x \in A \Rightarrow x \in B \Rightarrow$ أي $1 \leq 1$

$x \in B$ و $x \notin A \Rightarrow$ أي $0 \leq 1$

$x \notin A$ و $x \notin B \Rightarrow$ $0 \leq 0$

④ $\chi_{B \setminus A}(x) = \chi_B(x) - \chi_{A \cap B}(x) \quad ; A \subseteq B$



$B = A \cup B \setminus A$

$A \cap B \setminus A = \emptyset$

$$\chi_B(x) = \chi_A(x) + \chi_B(x)$$

Scanned by CamScanner

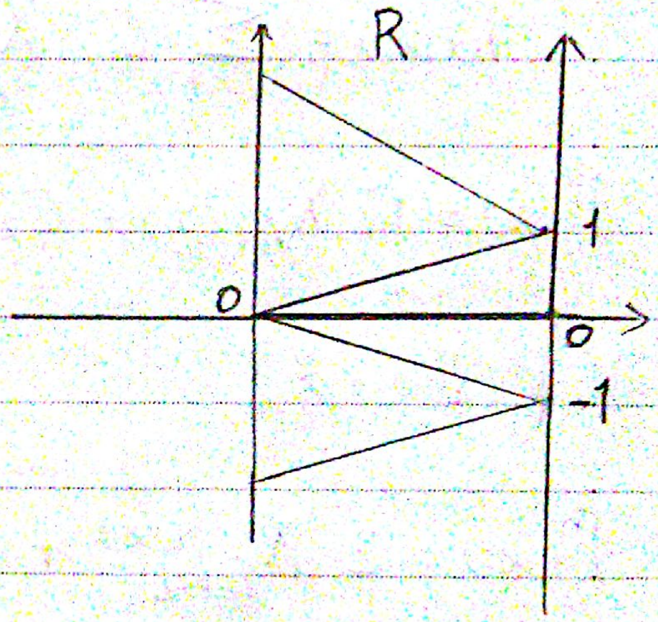
(5) $X_{A \setminus B} = X_A(x) - X_{A\cap B}(x)$

(6) $X_{A\cup B}(x) = X_A(x) + X_B(x) - X_{A\cap B}(x)$

(7) $X_{A\Delta B}(x) = X_A(x) + X_B(x) - 2X_{A\cap B}(x)$

التابع البسيط : (عبارة عن تابع مجموعة قيمه منتهية)

$$f : X \longrightarrow \{C_1, C_2, \dots, C_n\}$$



$$f(x) = \begin{cases} 1 & : \infty > x > 0 \\ 0 & : x = 0 \\ 1 & : -\infty < x < 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & : 0 \le x < 1 \\ 1 & ; 1 \le x < 2 \\ 2 & : 2 \le x < 3 \\ 3 & : x = 3 \end{cases}$$

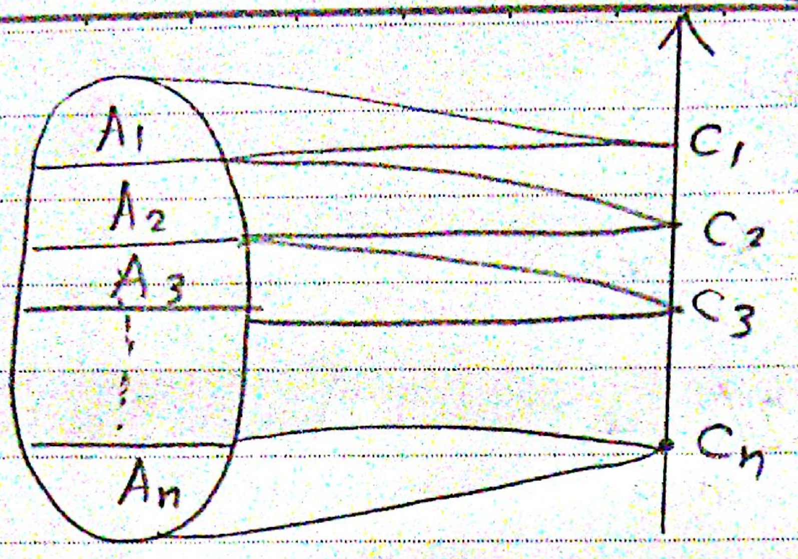
$[0, 1]$

$$f(x) = \begin{cases} 1 & : x \in Q \\ 0 & : x \notin Q \end{cases}$$

برهنة:

التابع البسيط يكتب كتركيب خطي لمجموعة من التوابع الدرجية بمعاملات التابع البسيط

$$\bigcup_{i=1}^{\infty} A_i = X \quad , \quad A_i \cap A_j = \phi \quad ; \ i \neq j$$



X

$$f(x)=\sum_{i=1}^{n} C_i \chi_{A_i}(x)$$

$$f(x)=C_1\chi_{A_1}(x)+C_2\chi_{A_2}(x)+\cdots+C_n\chi_{A_n}(x)$$

$$X\in[0,7]$$

$$f(x)=2\chi_{[0,1[}(x)+3\chi_{[3,6[}(x)+7\chi_{[6,7]}(x)$$

$$f(6)=7$$

$$f(x)=\begin{cases}2 & :[0,3[\\ 3 & :[3,6[\\ 7 & :[6,7]\end{cases}$$

تابع آخر:

$$g(x)=\begin{cases}5 & :[0,4[\\ 10 & :[4,7]\end{cases}$$

$$\Psi=f+g=(2+5)\chi_{[0,3[}+(3+5)\chi_{[3,4[}+(3+10)\chi_{[4,6[}$$

$$+(7+10)\chi_{[6,7]}$$

تعريف: $(X, \mathcal{A}, \mu)$ فضاءً قيوساً.
نعرف تكامل لوبيغ للتابع البسيط f بالنسبة للقياس μ على الفضاء X بالشكل:

$$\int_X f\, d\mu = \sum_{i=1}^{n} C_i\, \mu(A_i)$$

حيث C_i قيم التابع البسيط f و A_i تمثل تجزئة لـ X بحيث $A_i \cap A_j = \emptyset$ و $X = \bigcup_{i=1}^{n} A_i$: $i \neq j$

$$A_i = \{x \in X : f(x) = C_i\} = f^{-1}(\{C_i\})$$

$[0,1]$

$$f : \begin{cases} 0 & : x \in \mathbb{Q} \\ 1 & : x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

$$\left.\begin{aligned} A_1 &= \mathbb{Q} \\ A_2 &= \mathbb{Q}^c = [0,1] \setminus \mathbb{Q} \end{aligned}\right\} \begin{aligned} A_1 \cap A_2 &= \emptyset \\ A_1 \cup A_2 &= [0,1] \end{aligned}$$

$$\int_{[0,1]} f\, d\mu = 1 \cdot \mu(\mathbb{Q}) + 0\, \mu([0,1] \setminus \mathbb{Q})$$

قياس أي مجموعة عدودة $(\mathbb{Q}, \mathbb{N}, \mathbb{Z})$ يساوي الصفر

$$= 1 \cdot 0 + 0 \cdot 1 = 0$$

$[3] = [3,3]$ $\qquad \bigcup_{i=1}^{\infty} [x_i] = \mathbb{Q}$

$[x_i] = [x_i, x_i]$

$$\mu\left(\bigcup_{i=1}^{\infty} [x_i]\right) = \mu\left(\bigcup_{i=1}^{\infty} [x_i, x_i]\right)$$

$$= \sum_{i=1}^{\infty} \mu[x_i, x_i] = 0$$

$$[x] = \begin{cases} 0 & : 0 \le x < 1 \\ 1 & : 1 \le x < 2 \\ 2 & : 2 \le x < 3 \end{cases}$$

$$\int_{[0,3]} [x]\, d\mu = 0\mu[0,1[+ 1\mu[1,2[+ 2\mu[2,3[+ 3\mu[3,3]$$

$$= 0\cdot 1 + 1\cdot 1 + 2\cdot 1 + 3\cdot 0 = 3$$

$$\int_0^3 [x]\, dx = \int_1^2 dx + 2\int_2^3 dx$$

$$= [x]_1^2 + 2[x]_2^3 = 1 + 2 = 3$$

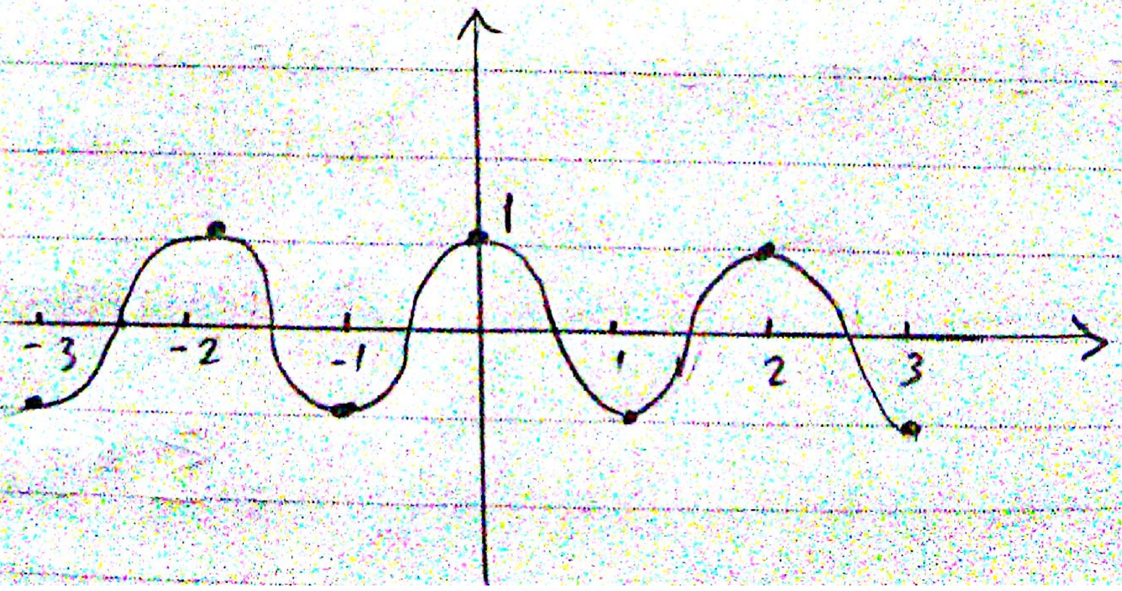
$$(R)\int_a^b f(x)\,dx \quad \Big\{ \quad S\int_a^b f\,dg \quad \Big\{ \quad L\int_{[a,b]} f\,d\mu$$ (تعميم R)

مثال:

$$L\int_{[-3,3]} \mathrm{sign}(\cos \pi x)\, d\mu$$

$$\mathrm{Sig}(w) = \begin{cases} 1 & : w > 0 \\ 0 & : w = 0 \\ -1 & : w < 0 \end{cases}$$

$$= C_1\mu(A_1) + C_2\mu(A_2) + C_3\,\mu(A_3)$$



$\cos \pi x = 0 \quad , \pi x = \frac{\pi}{2} + \pi k$

$x = \frac{1}{2} + k$

$k = -1 \Rightarrow x = -\frac{1}{2}$ | $k = 0 \Rightarrow x = \frac{1}{2}$

$k = -2 \Rightarrow x = -\frac{3}{2}$ | $k = 1 \Rightarrow x = \frac{3}{2}$

$k = -3 \Rightarrow x = -\frac{5}{2}$ | $k = 2 \Rightarrow x = \frac{5}{2}$

$A_2 = \{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2} \}$

$A_1 =]-\frac{5}{2}, -\frac{3}{2}[\cup]-\frac{1}{2}, \frac{1}{2}[\cup]\frac{3}{2}, \frac{5}{2}[$

$\mu(A_1) = 1 + 1 + 1 = 3$

$A_3 =]-3, -\frac{5}{2}[\cup]-\frac{3}{2}, -\frac{1}{2}[\cup]\frac{1}{2}, \frac{3}{2}[\cup]\frac{5}{2}, 3]$

$\mu(A_3) = \frac{1}{2} + 1 + 1 + \frac{1}{3} = 3$

$\int_{[-3,3]} \operatorname{sign}(\cos \pi x) \, d\mu = 1(3) + 0(0) - 1(3) = 0$