مراجعة عقدية

1 Review of complex numbers

This chapter contains a short review of complex numbers. You should know all of this material already from the core (G11LMA), so most of it will not be presented in the lectures.

1.1 Algebra of complex numbers

The complex numbers can be obtained from the real numbers by adding a new symbol, i, the *imaginary unit*, with the property that $i^2 = -1$.

A *complex number* is then an object $z = x + iy = x + yi$ that is formed from two real numbers x and y and the symbol i. The number x is called the *real part* of z and y is called the *imaginary part*. We write $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

We can define addition, subtraction, multiplication and division of complex numbers in the following way. Suppose that $z = x + iy$ and $w = u + iv$ are complex numbers (where u, v, x, y are all real). Then we set

$$\begin{aligned} z + w &= (x+u) + i(y+v), \\ z - w &= (x-u) + i(y-v), \\ zw &= (xu - yv) + i(xv + yu). \end{aligned}$$

If $z = x + iy \neq 0 + i0$, we also set

$$z^{-1} = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}$$

and so

$$\frac{w}{z} = wz^{-1} = \frac{xu + yv + i(xv - yu)}{x^2 + y^2}.$$

For the set of all complex numbers we write $\mathbb{C}$. The way we have defined our operations makes $\mathbb{C}$ into a field, which contains $\mathbb{R}$ since every real number x is the complex number $x + i0$.

Examples. With $z = 1 + i$ and $w = 1 - 2i$ we have

$$\begin{aligned} z + w &= (1+1) + (1-2)i = 2 - i \\ z - w &= (1-1) + (1-(-2))i = 3i \\ zw &= (1-(-2)) + (1-2)i = 3 - i \\ \frac{1}{z} &= \frac{1-i}{1+1} = \frac{1}{2} - \frac{1}{2}i \end{aligned}$$

تحذير ملاحظة

Warning. Note that the imaginary part is always a real number. The imaginary part of $2+3i$ is 3, not $3i$. If a and b are arbitrary complex numbers, the imaginary part of $a+ib$ is $\operatorname{Im} a + \operatorname{Re} b$, not just b.

عشوائية

مستوى تخيلي

1.2 The complex plane

نعرف

We usually identify the complex number $z = x + iy \in \mathbb{C}$, where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are both real, with the point $(x, y) \in \mathbb{R}^2$, the two dimensional plane, sometimes called *Argand plane* or *Argand diagram*. Real numbers x correspond to the points $(x, 0)$ on the *real axis*, purely imaginary numbers of the form iy with real y correspond to points $(0, y)$ on the *imaginary axis*.

تتعلق

(مستوى ثنائي البعد 2)

محور حقيقي

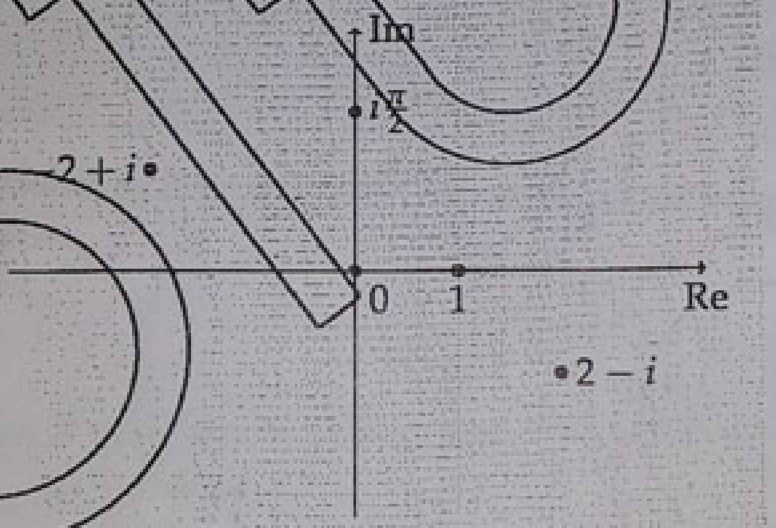


Figure 1: Some numbers in the complex plane

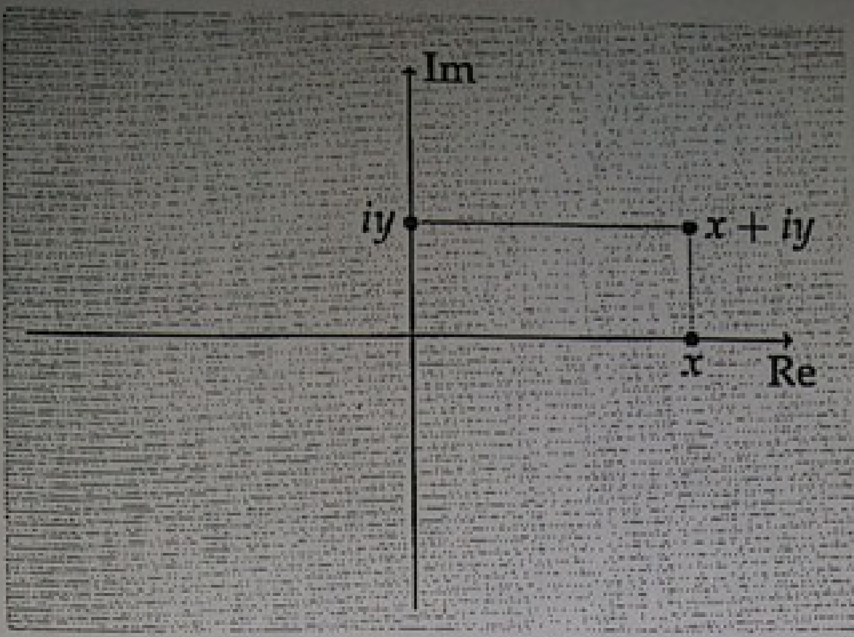
مستوى

المرافق

1.3 The complex conjugate

For a complex number z, we define its *complex conjugate*, written $\bar{z}$ as $\bar{z} = \operatorname{Re} z - i \operatorname{Im} z$. In some books you may see the notation z^* used instead.

تستخدم بدلاً الترميز



الشكل ٢   قسمه الحقيقي   مجموع   وقسمه التخيلي

Figure 2: The complex number $x + iy$ is the sum of its real part x and the purely imaginary number iy, which is i times its imaginary part.

والذي هو (i) مرة من قسمه التخيلي

Examples. أمثلة

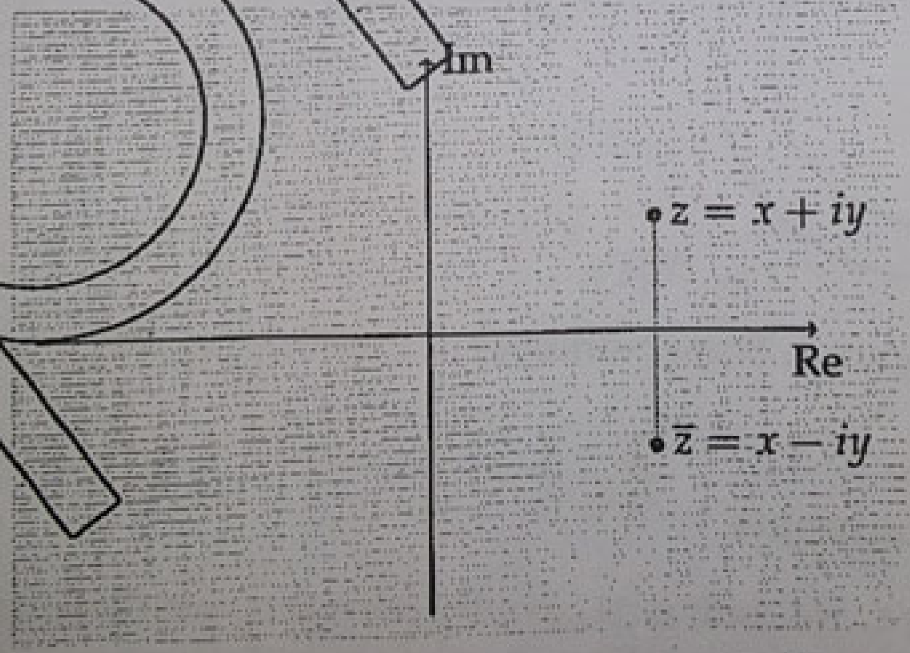
- $\overline{-2-i} = -2+i$
- $\overline{\cos t + i \sin t} = \cos t - i \sin t$ for $t \in \mathbb{R}$
- $\overline{2} = 2$
- $\overline{i\pi} = -i\pi$.

هندسيا   المحور الحقيقي

Geometrically, complex conjugation can be understood as reflection at the real axis, see Figure 1.3.

الشكل   انعكاس ؟ يمكن ان يكون مفهوما
أو نظير

نظيره
بالنسبة
المحور x



على - عبر

Figure 3: Complex conjugation is reflection across the real axis

يمكن التحقق منها بسهولة

التالية

له الخصائص المحققة بشكل سهل

المرافق

The conjugate has the following easily verified properties:

$$\overline{\overline{z}} = z, \quad \overline{z+w} = \overline{z} + \overline{w}, \quad \overline{zw} = \overline{z}\,\overline{w}, \quad z + \overline{z} = 2\,\mathrm{Re}\,z, \quad z - \overline{z} = 2i\,\mathrm{Im}\,z.$$

العلاقات المفيدة التالية

بالتحديد In particular we have the following useful relations:

$$\mathrm{Re}\,z = \frac{1}{2}(z + \overline{z}), \quad \mathrm{Im}\,z = \frac{1}{2i}(z - \overline{z}).$$

طويلة

1.4 Modulus of a complex number

القيمة المطلقة

حيث

The *modulus* or *absolute value* of a complex number $z = x + iy$ (where $x, y \in \mathbb{R}$), is defined as تعرف كالتالي

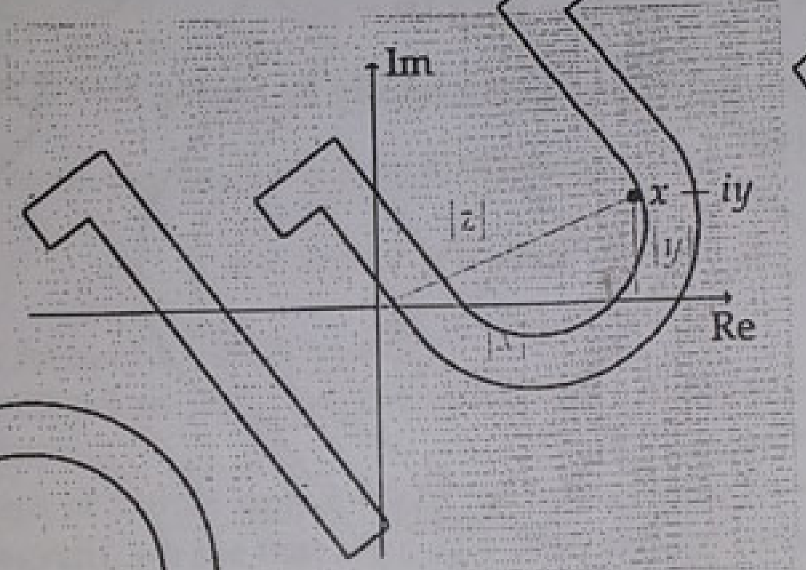
$$|z| = \sqrt{(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2} = \sqrt{x^2 + y^2}.$$

هندسيا المسافة النقطة ح من بشكل واضح

Geometrically, $|z|$ is the distance of the point z from 0 in the complex plane. It obviously satisfies يحقق

$$|x| \le |z|, \quad |y| \le |z|, \quad |z| \le |x| + |y|.$$

محققة فوراً لكون المثلث قائم



الشكل المسافة محسوبة باستخدام فيثاغورث

Figure 4: The modulus is the distance of z to zero, calculated using Pythagoras' theorem: $|z|^2 = x^2 + y^2$ نظرية

لاحظ أن: Note that

$$z\overline{z} = (\mathrm{Re}\,z + i\,\mathrm{Im}\,z)(\mathrm{Re}\,z - i\,\mathrm{Im}\,z) = (\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2 = |z|^2$$

الصيغة المفيدة

so that a useful formula is $|z| = \sqrt{z\overline{z}}$. Also we have

(i) $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$ if $z \neq 0$;

(ii) $|zw| = |z||w|$

لاثبات

To prove (ii) we write

العبارة

$$|zw|^2 = zw \cdot \overline{zw} = zw\overline{z}\overline{w} = |z|^2|w|^2.$$

تحذيرات قواعد صحيحة فقط

Warnings: (i) The rules $|z| = \pm z, z^2 = |z|^2$, are only true if z is real;
(ii) The statement $z < w$ is meaningless unless z and w are both real: you cannot compare non-real complex numbers this way.

لا يمكن المقارنة بهذه الطريقة بدون معنى ما لم تكن z, w كلاهما حقيقية

(متراجحة المثلث)

1.5 The triangle inequality

For all $z, w \in \mathbb{C}$, we have the *triangle inequality*

$$|z+w| \le |z| + |w|$$

عكس

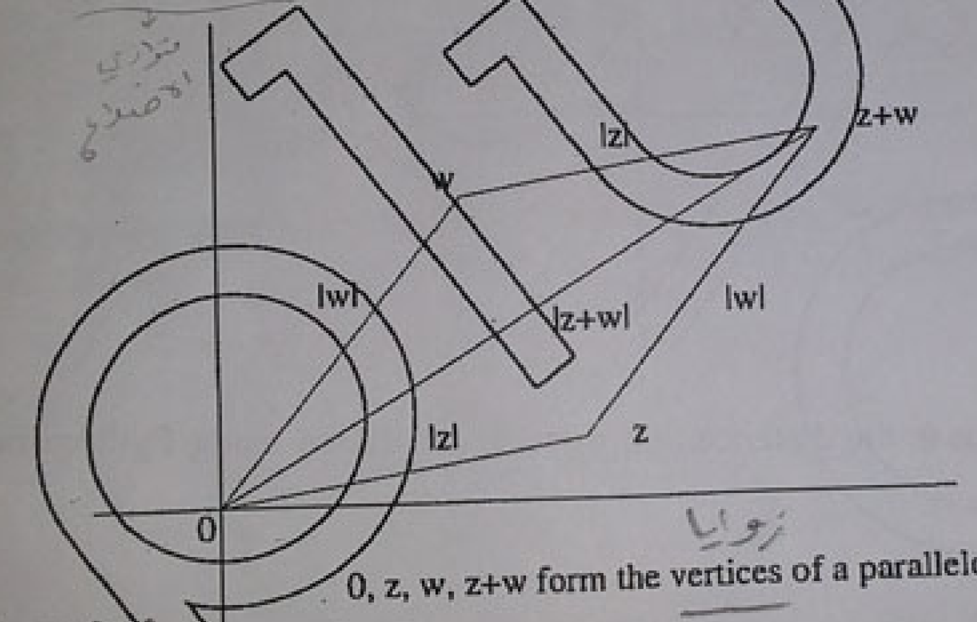
and the *reverse triangle inequality*

$$|z-w| \ge |z| - |w|.$$

بملاحظة

The first of these is most easily seen geometrically by noting that $0, z, w, z+w$ form the vertices of a parallelogram.

تشكل تكون رؤوس متوازي الاضلاع



زوايا

0, z, w, z+w form the vertices of a parallelogram

الاثبات الجبري كما يلي

An algebraic proof is as follows:

$$|z+w|^2 = (z+w)(\overline{z}+\overline{w}) = |z|^2 + z\overline{w} + \overline{z}w + |w|^2$$

نضع ومن هنا

Set $\zeta = z\overline{w}$ so $2\operatorname{Re}\zeta = \zeta + \overline{\zeta} = z\overline{w} + \overline{z}w$. Then $\operatorname{Re}\zeta \le |\zeta| = |z||\overline{w}| = |z||w|$ and hence

يتطلب البرهان

$$|z+w|^2 \le |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2$$

مزعوم

so $|z+w| \le |z| + |w|$ as claimed. The reverse inequality follows from the usual one by noting $|z| = |w + z - w| \le |w| + |z - w|$.

بملاحظة

معنى ذلك: لاثبات العكس نطلق من المتراجحة الاصلية في الاثبات الاول

بملاحظة ان

ما = |z|

شكل + أسي قطبي

1.6 Polar and exponential form

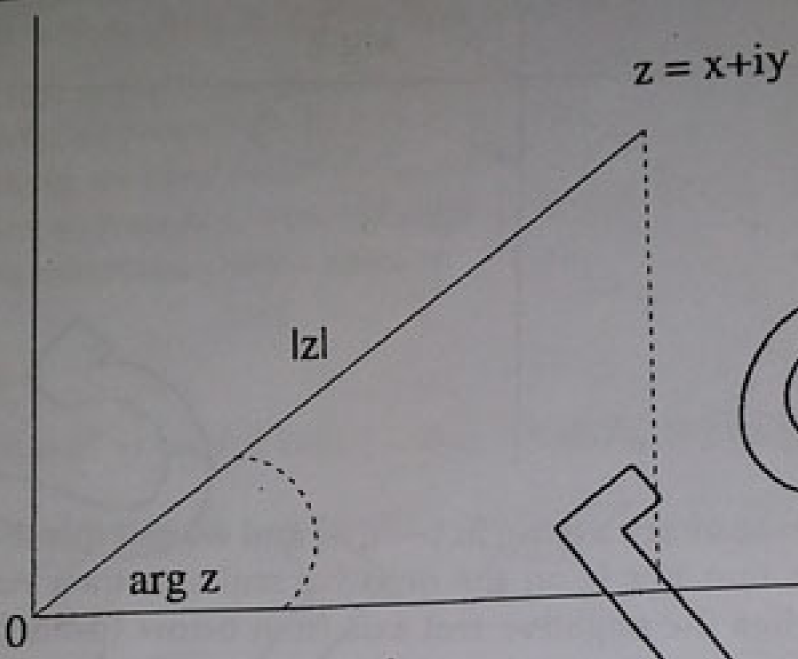
لنربط

Associate the complex number z with the point (Re (z), Im (z)) in $\mathbb{R}^2$.
If $z \neq 0$, then $x = \text{Re}(z)$ and $y = \text{Im}(z)$ are not both zero, and $r = |z| \neq 0$. Let θ be the angle between the positive real axis and the line from 0 to z, measured counter-clockwise in radians.

زاوية

عكس عقارب الساعة

بالراديان محور مستقيم المقاس



Then $x = \text{Re}(z) = r\cos\theta$, $y = \text{Im}(z) = r\sin\theta$. Writing

$$z = r\cos\theta + ir\sin\theta,$$

we have the POLAR form of z.

صيغة قطبي

The real number θ is called an ARGUMENT of z and we write $\theta = \arg z$. Note that:

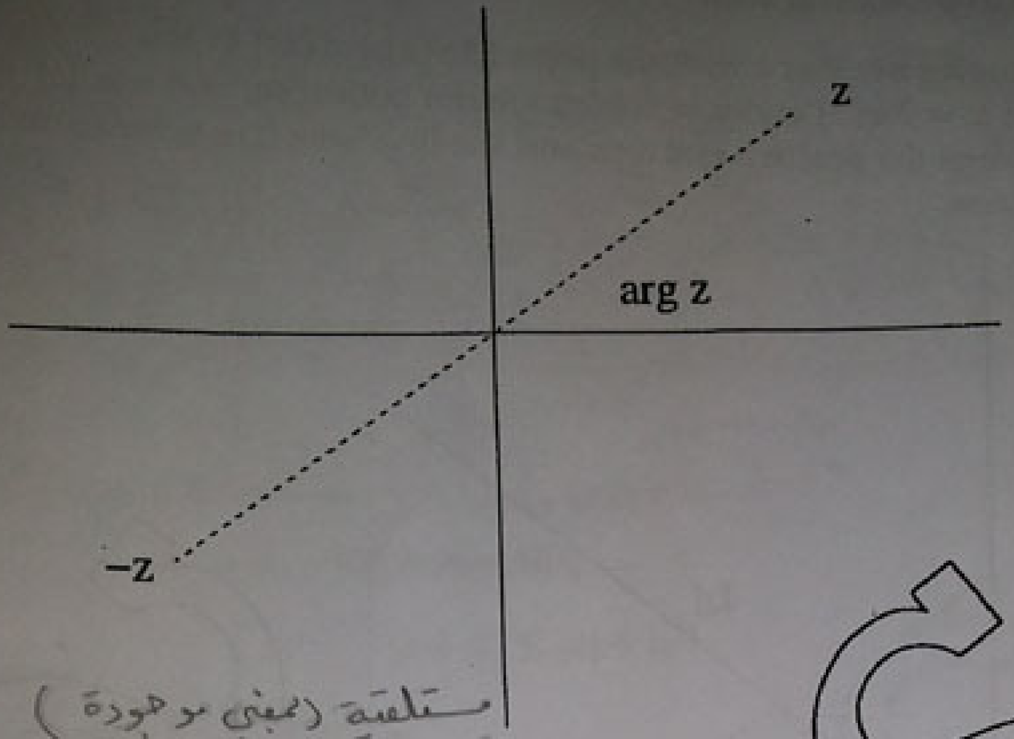
(1) arg 0 does not exist;

(2) If θ is one argument of z, then so is $\theta + k2\pi$ for any integer k;

From the Argand diagram, we see that $\arg z \pm \pi$ is an argument of $-z$.

غير موجود عدد صحيح

أرغاند مخطط القيمة المعكوسة



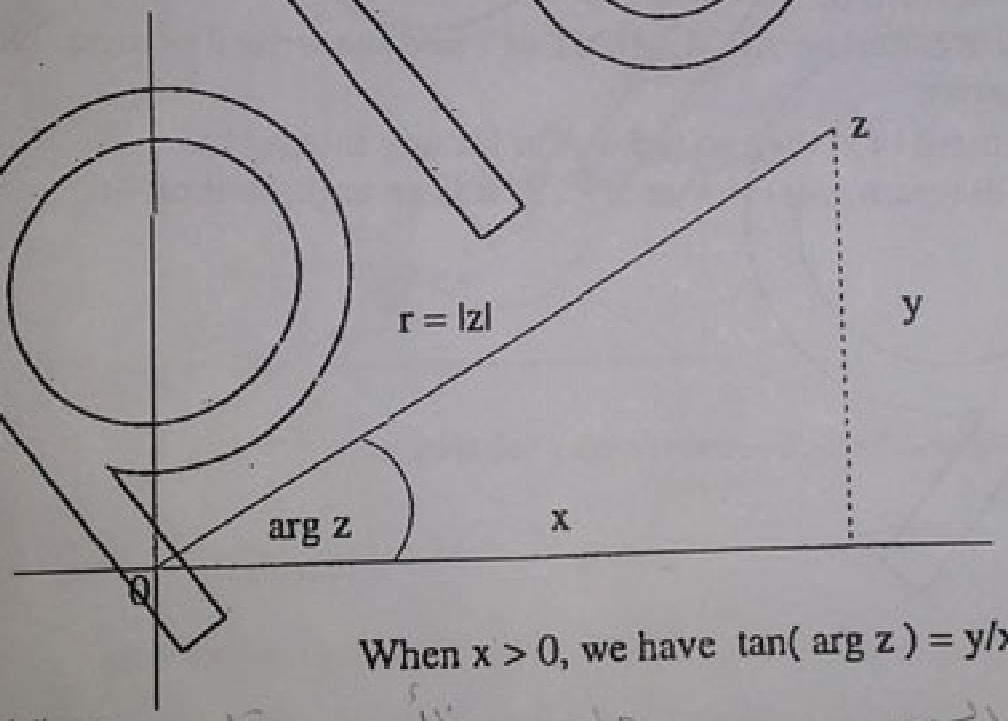
متعلقة (معنى موجودة) القيمة الرئيسية

We can always choose a value of arg z lying in $(-\pi, \pi]$ and we call this the PRINCIPAL ARGUMENT Arg z. Note that if z is on the negative real axis then Arg z $= \pi$, but Arg z $\to -\pi$ as z approaches the negative real axis from below (from the lower half-plane).

مختار الاقتراب الخلف الأدنى نصف المستوى

حساب باستخدام الحاسبة نفرض

How to compute Arg z using a calculator: suppose $z = x + iy \neq 0$, with x, y real. If $x > 0$ then $\theta = \text{Arg}\, z = \tan^{-1}(y/x) = \arctan(y/x)$.



When x > 0, we have tan(arg z) = y/x .

هذه الطريقة تعطي جواب خطأ لأن هو السبب الآلة الحاسبة

However this method gives the WRONG answer if $x < 0$. The reason is that calculators always give arctan between $-\pi/2$ and $\pi/2$. In fact if $x < 0$ then

في الواقع نفرض

$$\arctan(y/x) = \arctan(-y/(-x)) = \text{Arg}\,(-z),$$

and this is the same as $\operatorname{Arg} z \pm \pi$. If $x = 0$ and $y > 0$ then $\operatorname{Arg} z = \pi/2$, while if $x = 0$ and $y < 0$ then $\operatorname{Arg} z = -\pi/2$.

Exercise: let $0 \leq \alpha \leq \pi/2$ and $0 \leq \beta \leq \pi/2$.
Find a formula for the reflection of each z across the straight line L making an angle α with the positive real axis.
If we do this first with angle α, and then with angle β, what is the net effect?

Solution. We first rotate L into the real axis by $z \mapsto e^{-i\alpha}z$. Then we reflect across the real axis by $z \mapsto \bar{z}$. Finally, we rotate back via $z \mapsto e^{i\alpha}z$.
Combining everything, we have $z \mapsto e^{i\alpha}\overline{e^{-i\alpha}z} = e^{2i\alpha}\bar{z}$.

If we do this first with angle α, then with angle β, we obtain $z \mapsto e^{2i\beta}\overline{e^{2i\alpha}\bar{z}} = e^{2i(\beta-\alpha)}z$, so the combination of two reflections yields a rotation.

1.7 Definition

تعريف

صيغ المثلثات

For t real, we define $e^{it} = \cos t + i \sin t$. Using the trigonometric formulas

نحصل على

$$\cos(s+t) = \cos s \cos t - \sin s \sin t, \quad \sin(s+t) = \sin s \cos t + \cos s \sin t,$$

we get, for s, t real,

$$e^{is}e^{it} = \cos s \cos t - \sin s \sin t + i(\cos s \sin t + \sin s \cos t) = e^{i(s+t)}.$$

Thus $e^{-it}e^{it} = e^{i0} = 1$. Also $\overline{(e^{it})} = e^{-it}$ and, if z, w are non-zero complex numbers, we have

$$zw = |z|e^{i \arg z}|w|e^{i \arg w} = |zw|e^{i(\arg z + \arg w)}$$

(so we can multiply two complex numbers by multiplying their absolute values and adding the arguments) and

$$\bar{z} = |z|e^{-i \arg z}, \quad \frac{1}{z} = |z|^{-1}e^{-i \arg z}.$$

We get:
(a) $\arg z + \arg w$ is an argument of zw. (b) $-\arg z$ is an argument of $1/z$ and of $\bar{z}$.
تحذير **Warning:** it is not always true that $\operatorname{Arg} z + \operatorname{Arg} w = \operatorname{Arg} zw$. Try $z = w = -1 + i$.

مسألة **Problem:** for s, t real, when is e^{it} equal to 1? When do we have $e^{is} = e^{it}$?

For $e^{it} = 1$ we need

$$1 = \cos t + i \sin t,$$

which occurs if and only if $\cos t = 1$ (because this gives $\sin t = 0$ also). Hence $e^{it} = 1$ if and only if t is an integer times 2π.

Next, $e^{is} = e^{it}$ if and only if $e^{i(s-t)} = 1$ i.e. $s - t$ is an integer times 2π.

نظرية دي موفير

1.8 De Moivre's theorem

For t real, we have

$$e^{2it} = e^{it}e^{it} = (e^{it})^2, \quad e^{-it} = \frac{1}{e^{it}}.$$

باعادة المناقشة نحصل

Repeating this argument we get $(e^{it})^n = e^{int}$ for all real t and integer n (this is called de Moivre's theorem).

For example, for real t, we have

$$\cos 2t = \mathrm{Re}\,(e^{2it}) = \mathrm{Re}\,((e^{it})^2) = \mathrm{Re}\,(\cos^2 t - 2i\cos t \sin t - \sin^2 t) = 2\cos^2 t - 1.$$

واحدة جذور

1.9 Roots of unity

صحيح حلول

Problem: let n be a positive integer. Find all solutions z of $z^n = 1$.

بشكل واضح الحل

Solution: clearly $z \neq 0$ so we write $z = re^{it}$ with $r = |z|$ and t an argument of z. Then $1 = z^n = r^n e^{int}$. So $1 = |z^n| = r^n$ and $r = 1$.

Next, we have $e^{int} = \cos nt + i \sin nt = 1$. From our discussion in §1.7 this gives $nt = k2\pi$ for some integer k, and so $z = e^{it} = e^{k2\pi i/n}$. However, also from §1.7, we have $e^{k2\pi i/n} = e^{k'2\pi i/n}$ if and only if $(k - k')2\pi/n$ is an integer multiple of 2π i.e. $k - k'$ is an integer multiple of n. So we just get the n roots $\zeta_k = e^{k2\pi i/n}, k = 0, 1, \ldots, n-1$. One of them (given by $k = 0$) is 1, and the n roots are equally spaced around the circle of centre 0 and radius 1, each adjacent pair separated by an angle $2\pi/n$. The ζ_k are called the n'th roots of unity.

Example: for $n = 8$ the roots of unity are given by

$$e^{i0} = 1, \quad e^{i\pi/4} = \frac{1+i}{\sqrt{2}}, \quad e^{i\pi/2} = i, \quad e^{3i\pi/4} = \frac{-1+i}{\sqrt{2}},$$

and

$$e^{i\pi} = -1, \quad e^{5i\pi/4} = \frac{-1-i}{\sqrt{2}}, \quad e^{3i\pi/2} = -i, \quad e^{7i\pi/4} = \frac{1-i}{\sqrt{2}}.$$

يمكن أن
يأتي مثل
هذا بالامتحان

$z^n = e^{int} = 1$

$e^{int} = e^{(2\pi k)i}$

$t = \frac{2\pi k}{n}$

$k = 0, 1, \ldots, n-1$

$z^8 = r(e^{it})^8$

$\boxed{r=1}$

$z^8 = (e^{it})^8 = 1$

$8t = 2\pi k$

$t = \frac{2\pi k}{8}$

$k = 0, \ldots$

$k = 0$

$t = 0 \Rightarrow z_1 = e^{i0} = 1$

$r^n e^{int} = (re^{it})^n = z^n = w = |w| e^{i\,\text{Arg}\, w}$

$z = |w|^{\frac{1}{n}} e^{i\frac{\text{Arg}\, w\, 2\pi k}{n}}$ $k = 0, \dots, n-1$

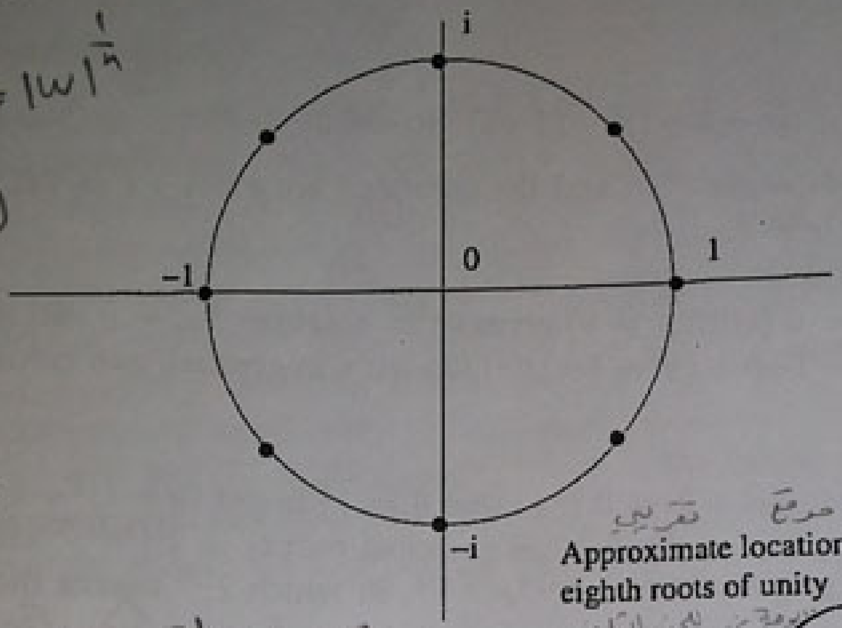
$r^n e^{int} = |w| e^{i\,\text{Arg}\, w}$

$r^n = |w| \Rightarrow r = |w|^{\frac{1}{n}}$

$nt = \text{Arg}\, w (2\pi k)$

$t = \frac{\text{Arg}\, w(2\pi k)}{n}$

$k = 0, 1, \dots, n-1$



موقع تقريبي

Approximate location of the eighth roots of unity

المعادلات البسيطة بعض حل

1.10 Solving some simple equations

To solve $z^n = w$, where n is a positive integer and w is a non-zero complex number, we first write $w = |w|e^{i\text{Arg}\, w}$. Now

(الجذر الاول)

$$z_0 = |w|^{1/n} e^{i(\text{Arg}\, w)/n},$$

تدل

in which $|w|^{1/n}$ denotes the positive n'th root of $|w|$, gives

$$(z_0)^n = |w| e^{i\text{Arg}\, w} = w,$$

تحقق نظرية ديموفر الجذر الرئيسي

using de Moivre's theorem. This root z_0 is called the principal root.

Next we find all other roots. If z is any root of $z^n = w$, then

$$\left(\frac{z}{z_0}\right)^n = \frac{z^n}{z_0^n} = \frac{w}{w} = 1,$$

الصيغة

and so z/z_0 is an n'th root of unity as given by the formula in §1.9. So the n roots of $z^n = w$ are

$$z_k = z_0 e^{k2\pi i/n} = |w|^{1/n} e^{i(\text{Arg}\, w)/n + k2\pi i/n}, \quad k = 0, 1, \dots, n-1.$$

حل

For example, to solve $z^4 = -1 - i = w$, we write $w = \sqrt{2}e^{-3\pi i/4}$ and $z_0 = 2^{1/8}e^{-3\pi i/16}$. The other roots are $z_1 = 2^{1/8}e^{-3\pi i/16 + \pi i/2} = 2^{1/8}e^{5\pi i/16}$ and $z_2 = 2^{1/8}e^{-3\pi i/16 + \pi i} = 2^{1/8}e^{13\pi i/16}$ and $z_3 = 2^{1/8}e^{-3\pi i/16 + 3\pi i/2} = 2^{1/8}e^{-3\pi i/16 - \pi i/2} = 2^{1/8}e^{-11\pi i/16}$.

التربيع

1.11 Quadratics

مربع الطريقة الاعتيادية حل

We solve these by completing the square in the usual way. For example, to solve

اكمال

$$z^2 + (2 + 2i)z + 6i = 0$$

we write this as

$$(z+1+i)^2-(1+i)^2+6i=0$$

giving $(z+1+i)^2 = -4i = 4e^{-i\pi/2}$ and the solutions are $z+1+i = 2e^{-i\pi/4}$ and $z+1+i = 2e^{-i\pi/4+i\pi} = 2e^{3i\pi/4}$.

الحلول

In general, $az^2+bz+c=0$ (with $a \neq 0$) gives $4a^2z^2+4abz+4ac=0$ and so $(2az+b)^2 = b^2-4ac$. Hence $z = (-b+(b^2-4ac)^{1/2})/2a$ with, in general, two values for the square root.

For example, to solve $z^4-2z^2+2=0$ we write $u=z^2$ to get $(u-1)^2+1=0$ and so $u=1\pm i$. Now $z^2 = 1+i = \sqrt{2}e^{i\pi/4}$ has principal root $z_1 = 2^{1/4}e^{i\pi/8}$ and second root $z_2 = z_1e^{i\pi} = -z_1 = 2^{1/4}e^{i9\pi/8} = 2^{1/4}e^{-i7\pi/8}$, in which $2^{1/4}$ means the positive fourth root of 2. Two more solutions come from solving $z^2 = 1-i = \sqrt{2}e^{-i\pi/4}$ and these are $z_3 = 2^{1/4}e^{-i\pi/8}$ and $z_4 = 2^{1/4}e^{i7\pi/8}$.

تطبيقات تكامل

2 Integration and contours

مقدمة التكاملات العقدية

2.1 Introduction to complex integrals

مجال

Suppose first of all that $[a,b]$ is a closed interval in $\mathbb{R}$ and that the function $g : [a,b] \to \mathbb{C}$ is continuous (this means simply that $u = \text{Re}\,(g)$ and $v = \text{Im}\,(g)$ are both continuous). We can just define

(ماذا يعني تابع عقدي مستمر)

$$\int_a^b g(t)dt = \int_a^b \text{Re}\,(g(t))\,dt + i\int_a^b \text{Im}\,(g(t))\,dt.$$

Example: determine $\int_0^2 e^{2it}dt$.

Note that every complex number z can be written in the form $z = re^{iT}$ with $r = |z| \geq 0$ and $T \in \mathbb{R}$ and so $|z| = ze^{-iT}$. Thus we have, for some real T,

هام

$$\left|\int_a^b g(t)dt\right| = e^{-iT}\int_a^b g(t)dt = \int_a^b e^{-iT}g(t)\,dt = \int_a^b \text{Re}\,(e^{-iT}g(t))\,dt + i\int_a^b \text{Im}\,(e^{-iT}g(t))\,dt.$$

Since the modulus is a real number we get

$$\left|\int_a^b g(t)\,dt\right| = \int_a^b \text{Re}\,(e^{-iT}g(t))\,dt \leq \int_a^b |\text{Re}\,(e^{-iT}g(t))|\,dt \leq \int_a^b |e^{-iT}g(t)|\,dt = \int_a^b |g(t)|\,dt.$$

ممكن أن يأتي تابع آخر فنستفيد من الخاصية:

$$\underbrace{\left|\int_a^b g(t)\,dt\right|}_{\text{يسعى لـ } \infty} \leq \underbrace{\int_a^b |g(t)|\,dt}_{\text{يؤول إلى}}$$

إلى ∞ غالباً

Example: for $n \in \mathbb{N}$ set

$$\boxed{\begin{gathered} I_n = \int_1^2 \frac{e^{it^3}}{t+in}\,dt. \\ I_n \to 0 \\ n \to \infty \end{gathered}}$$

هام

Show that $I_n \to 0$ as $n \to +\infty$.

[illegible]

$$\left|\frac{e^{it^3}}{t+in}\right| = \underbrace{|e^{it^3}|}_{1}\left|\frac{1}{t+in}\right|$$

$$= \left|\frac{t-in}{t^2+n^2}\right| = \sqrt{\left(\frac{t}{t^2+n^2}\right)^2+\frac{n^2}{(t^2+n^2)^2}} = \frac{1}{\sqrt{t^2+n^2}}$$

ومنه: $\int_1^2 \left|\frac{e^{it^3}}{t+in}\right|dt \xrightarrow[n\to\infty]{} 0$

محيطات مسارات

تتبع خارج

2.2 Paths and contours

Suppose that f_1, f_2 are continuous real-valued functions on a closed interval $[a,b] \subseteq \mathbb{R}$. As the "time" t increases from a to b, the point $\gamma(t) = f_1(t) + if_2(t)$ traces out a path in $\mathbb{C}$. A path in $\mathbb{C}$ is thus just a continuous function γ from a closed interval $[a,b]$ to $\mathbb{C}$, and saying that γ is continuous just means that its real and imaginary parts are continuous. In order to integrate functions along paths we will need to introduce a special type of path with good properties, defined as follows.

A smooth contour is given by a continuous function $\gamma : [a,b] \to \mathbb{C}$, with the property that the derivative γ' exists and is continuous and never 0 on $[a,b]$. Notice that if we write $\mathrm{Re}\,(\gamma) = \sigma, \mathrm{Im}\,(\gamma) = \tau$ then $(\sigma'(t), \tau'(t))$ is the tangent vector to the curve, and we are assuming that this varies continuously and is never the zero vector.

The length of the smooth contour γ is defined to be

أملس

$$|\gamma| = \int_a^b |\gamma'(t)|\,dt.$$

Examples:
(i) The circle of centre $a \in \mathbb{C}$ and radius $r > 0$ is described by

$$\gamma(t) = a + re^{it}, \quad 0 \leq t \leq 2\pi.$$

Here $|\gamma(t) - a| = |re^{it}| = r$ and as t increases from 0 to 2π the point $\gamma(t)$ moves once counter-clockwise around the circle.

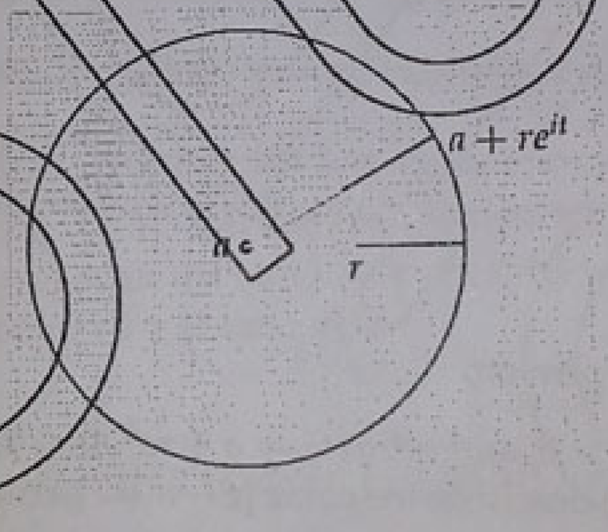


Figure 5: Circle around a of radius r

Moreover,

$$\gamma'(t) = ire^{it}, \quad |\gamma'(t)| = r,$$

طول

and so the length of the smooth contour is $2\pi r$ as expected.

كما متوقع

Note also that the modified formula

$$\gamma(t) = a + re^{-it}, \quad 0 \leq t \leq 2\pi,$$

مع عقارب الساعة

describes the same circle, but clockwise.

قطعة مستقيمة

(ii) For the straight line segment joining two points C, D ∈ C we can write

$$z = \gamma(t) = C + t(D - C), \quad 0 \le t \le 1.$$

ثابت

Here $\gamma'(t) = D - C$ (constant tangent) and the length of the contour is

$$\int_0^1 |D - C| dt = |D - C|$$

as expected (it is the distance between C and D).

اختياري

For more comments on paths and arc length see §2.10 (OPTIONAL!).

مقدمة تكاملات المسارات

2.3 Introduction to contour integrals

Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a smooth contour. If f is a function such that $f(\gamma(t))$ is continuous on $[a, b]$ we set

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

تفسير قاعدة السلسلة

We can interpret this simply as applying the chain rule for integrals: we are setting

$$z = \gamma(t), \quad dz = \gamma'(t)\, dt.$$

The condition that $f(\gamma(t))$ is continuous and the fact that $\gamma'(t)$ exists and is also continuous together ensure that the integral exists: this is why we defined smooth contours as we did.

وجود التكامل

2.4 Important example: powers

القوى

تزداد

Let $a \in \mathbb{C}$, let $m \in \mathbb{N}$ and $r > 0$, and set $\gamma(t) = a + re^{it}, 0 \le t \le 2m\pi$. As t increases from 0 to $2m\pi$, the point $\gamma(t)$ describes the circle $|z - a| = r$ counter-clockwise m times. Now let $n \in \mathbb{Z}$. We have

m مرة

$$\int_\gamma (z - a)^n\, dz = \int_0^{2m\pi} r^n e^{int} ire^{it}\, dt = \int_0^{2m\pi} ir^{n+1} e^{(n+1)it}\, dt.$$

If $n \neq -1$ this is

$$ir^{n+1} \int_0^{2m\pi} \cos(n+1)t + i \sin(n+1)t\, dt = \frac{ir^{n+1}}{n+1} [\sin(n+1)t - i\cos(n+1)t]_0^{2m\pi} = 0,$$

by the periodicity of $\cos((n+1)t)$ and $\sin((n+1)t)$. If $n = -1$ then we get $2m\pi i$.

دورية

This example will recur throughout the module!

2.5 Some properties of contour integrals

(a) If $\gamma : [a,b] \to \mathbb{C}$ is a smooth contour we can define a contour λ by $\lambda(t) = \gamma(b+a-t)$. Thus $\lambda(a) = \gamma(b), \lambda(b) = \gamma(a)$ and λ describes the same set of points as γ, but in the opposite direction. Then

ال تكامل عكسي

$$\int_\lambda f(z)\,dz = \int_a^b f(\gamma(b+a-t))(-\gamma'(b+a-t))\,dt$$

Using the substitution $s = b+a-t, dt = -ds$, we get

التبديل

$$\int_\lambda f(z)\,dz = \int_b^a f(\gamma(s))\gamma'(s)\,ds = -\int_\gamma f(z)\,dz.$$

تغير

Thus changing the direction has multiplied the integral by -1.

مر على نقطة اكثر من مرة

تعريف

(b) A smooth contour is called SIMPLE if it never passes through the same point twice i.e. it is a one-one (also called injective) function. Suppose that λ and γ are simple, smooth contours which describe the same set of points in the same direction. Suppose λ is defined on $[a,b]$ and γ on $[c,d]$. It is then easy to see that there is a strictly increasing function $\phi : [a,b] \to [c,d]$ such that $\lambda(t) = \gamma(\phi(t))$ for $a \le t \le b$. Here λ at time t passes through the same point as γ at time $\phi(t)$.

متزايد تماماً

We can then write

$$\int_\lambda f(z)\,dz = \int_a^b f(\lambda(t))\lambda'(t)\,dt = \int_a^b f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)\,dt.$$

The substitution $s = \phi(t)$ now gives

$$\int_\lambda f(z)\,dz = \int_c^d f(\gamma(s))\gamma'(s)\,ds = \int_\gamma f(z)\,dz.$$

مستقل معطاه

Thus the contour integral is "independent of parametrization".

Here's the proof that $\phi'(t)$ exists (OPTIONAL!). For t and t_0 in (a,b) with $t \ne t_0$ write

$$\frac{\lambda(t)-\lambda(t_0)}{t-t_0} = \frac{\gamma(\phi(t))-\gamma(\phi(t_0))}{\phi(t)-\phi(t_0)}\frac{\phi(t)-\phi(t_0)}{t-t_0}.$$

Note that there is no danger of zero denominators here as ϕ is strictly increasing so that $\phi(t) \neq \phi(t_0)$. Letting t tend to t_0 we have $\lambda(t) \to \lambda(t_0)$ and so $\gamma(\phi(t)) \to \gamma(\phi(t_0))$. We also have $\phi(t) \to \phi(t_0)$ (otherwise the increasing function ϕ must miss out some values in $[c,d]$ and so $\lambda(t)$ misses some points through which γ passes, since γ is one-one). Thus we see that

$$\phi'(t_0) = \lim_{t \to t_0} \frac{\phi(t) - \phi(t_0)}{t - t_0} = \frac{\lambda'(t_0)}{\gamma'(\phi(t_0))}$$

which gives the expected formula for ϕ' (*and* shows that it is continuous).

(c) This is called the FUNDAMENTAL ESTIMATE; suppose that $|f(z)| \leq M$ on γ. Then we have

مهم

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt \leq M \int_a^b |\gamma'(t)|\,dt = M|\gamma|,$$

where $|\gamma|$ denotes the length of γ.

Warning: It is *not* true (but a popular mistake) to write that

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)|\,dz;$$

the expression on the right hand side is usually not even a real number.

Example: let γ be the straight line from 2 to $3+i$, and let

$$I_n = \int_\gamma \frac{dz}{z^n - \bar{z}}$$

with n a positive integer. Show that $I_n \to 0$ as $n \to \infty$.

$2 \leq |z| \leq |3+i| = \sqrt{10}$

$|z^n - \bar{z}| \geq |z^n| - |z|$

$|z^n - \bar{z}| \geq 2^n - \sqrt{10}$

Solution. We have $\sqrt{10} \geq |3+i| \geq |z| \geq 2$ for all z on the contour γ, so (using the reverse triangle inequality) $|z^n - \bar{z}| \geq |z^n| - |z| \geq 2^n - \sqrt{10}$. We also have $|\gamma| = |3+i-2| = |1+i| = \sqrt{2}$. The fundamental estimate yields for $n \geq 2$

$f(z) = \dfrac{1}{|z^n - \bar{z}|}$

$\leq \dfrac{1}{2^n - \sqrt{10}} \to 0$

$$|I_n| \leq \frac{1}{2^n - \sqrt{10}}\sqrt{2},$$

and the right hand side clearly tends to 0 as $n \to \infty$, so $I_n \to 0$.

$|\gamma| = |3+i-2|$

$= |1+i| = \sqrt{2}$

قطعياً

2.6 Piecewise smooth contours

منحنى

By a PIECEWISE SMOOTH contour γ we mean finitely many smooth contours γ_k joined end to end, in which case we define

$$\int_\gamma f(z)dz = \sum_k \int_{\gamma_k} f(z)dz.$$

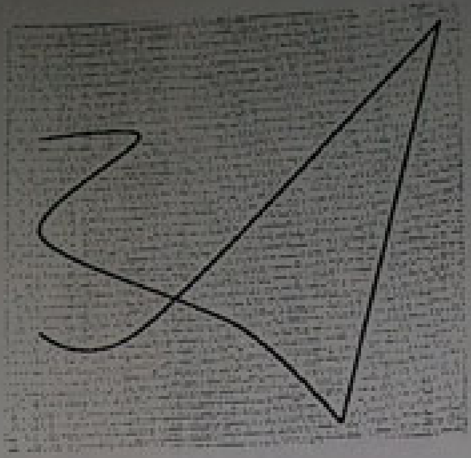


Figure 6: A piecewise smooth contour

For example, to go from 0 to $1+i$ via 1 along straight lines we can use $\gamma_1(t)=t, 0 \leq t \leq 1$, followed by $\gamma_2(t)=1+(1+i-1)t, 0 \leq t \leq 1$.

Note that if we need to compute $\int_\gamma f(z)dz$ then by §2.5(b) it does not matter which parametrization of the line segments we use. وضع المعايير

مضاعفة المنحنيات الملساء قطعيا

2.7 Properties of piecewise smooth contours

Suppose that γ is a PSC made up of the smooth contours $\gamma_1, \ldots, \gamma_n$, in order. تتشكل

تعريف (a) γ is SIMPLE if it never passes through the same point twice (apart from the fact that each γ_{k+1} starts where γ_k finishes).

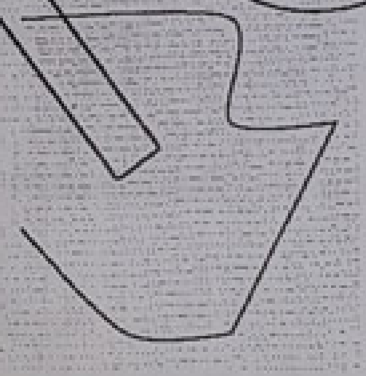


Figure 7: A simple piecewise smooth contour

انتهى من المكان الذي بدأ منه

تعريف (b) γ is CLOSED if it finishes where it started i.e. the last point of γ_n is the first point of γ_1.

ليس له علاقة

N.B. this has nothing to do with closed sets (which do not feature in G12COF),

تعريف (c) γ is called SIMPLE CLOSED if it finishes where it started but otherwise does not pass through any point twice (apart again from the fact that γ_{k+1} starts where γ_k finishes). Example:

Let σ be the straight line segment from i to 1, and let γ be the line segment from i to 0,

Figure 8: A closed piecewise smooth contour

Figure 9: A simple closed piecewise smooth contour

followed by that from 0 to 1. Show that

$$\int_{\gamma} \bar{z}\, dz \neq \int_{\sigma} \bar{z}\, dz.$$

Thus the contour integral is not always independent of path (we will return to this important theme later).

كموضوع مجال (منطقات) المجموعات ، المجالات المفتوحة متوضعة داخل

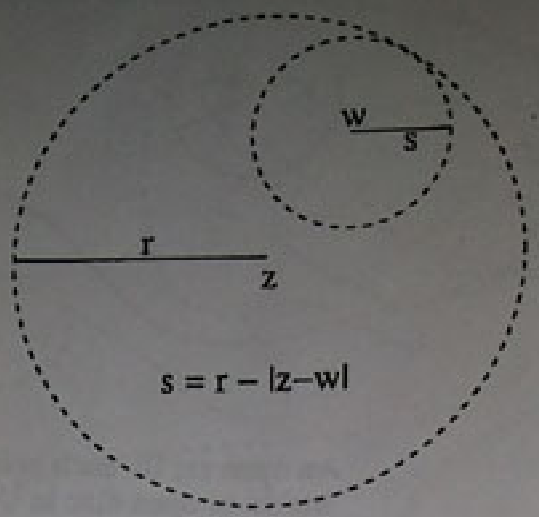
2.8 Open Sets and Domains

Let z ∈ C and let $r > 0$. We define $B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$. This is called the open disc of centre z and radius r. It consists of all points w lying inside the circle of centre z and radius r, the circle itself being excluded.

Suppose w is in the open disc $B(z,r)$. Then we can find an open disc of centre w which is contained in $B(z,r)$. To do this put $s = r - |w - z| > 0$. Then $B(w,s) \subseteq B(z,r)$. What we've done is to inscribe a circle of radius s and centre w within the circle of centre z and radius r.

رسم

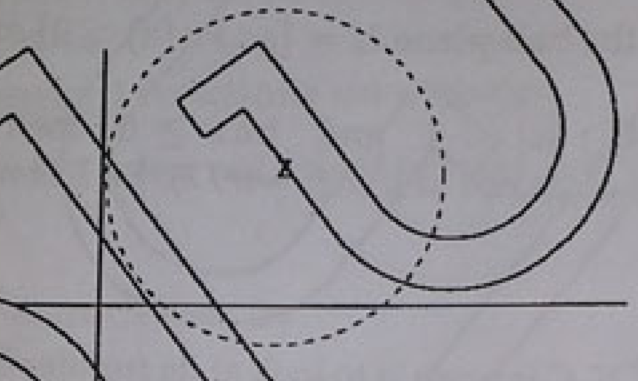
لوصف



We can also note that if $|u-w| < s$ then $|u-z| \leq |u-w| + |w-z| < s + |w-z| = r$ so $u \in B(w,s)$ implies $u \in B(z,r)$.

مشابه نفس المستوى

(ii) We can do something similar for a half-plane $H = \{z : \mathrm{Re}(z) > 0\}$. If $z \in H$, put $r_z = \mathrm{Re}\,(z) > 0$. Then $B(z, r_z) \subseteq H$.



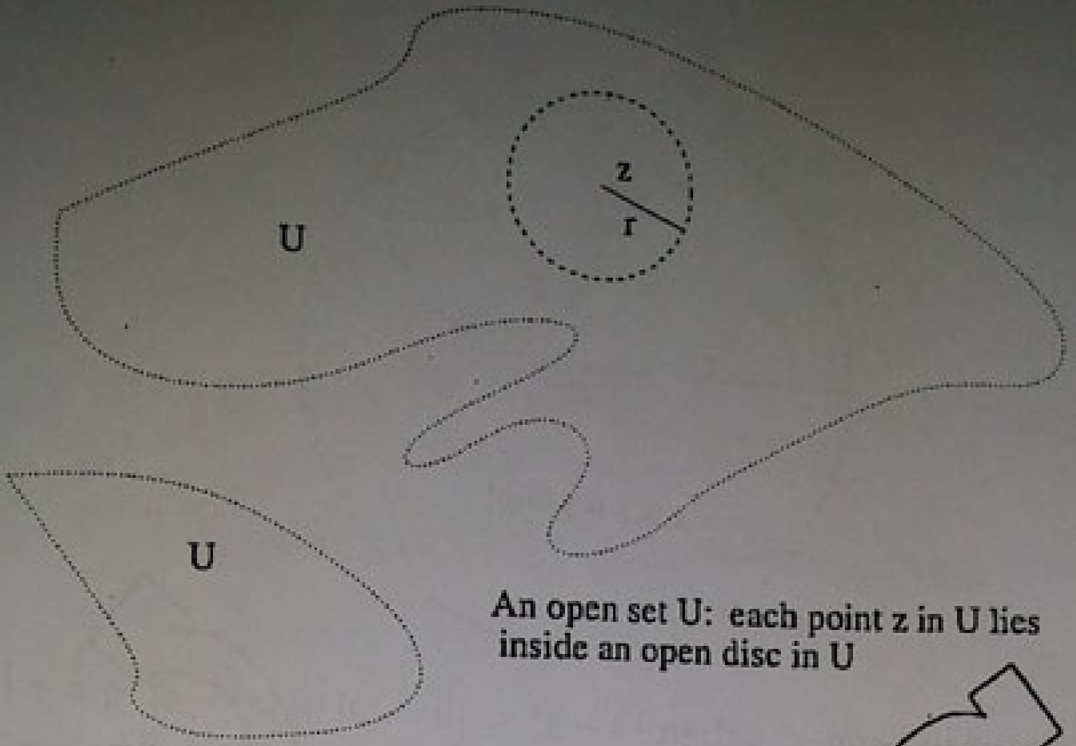
The right halfplane Re(z) > 0
Take radius Re(z).

الخاصية الانفتاح

These two examples have a property called *openness*. Let $U \subseteq \mathbb{C}$. We say that U is OPEN if it has the following property: for each $z \in U$ there exists $r = r_z > 0$ such that $B(z, r_z) \subseteq U$. Note that r_z will usually depend on z.

In the terminology of G12MAN (which will *not* be needed in G12COF) this says that every point in U is an interior point of U.

مصطلح داخلية مستعمل



An open set U: each point z in U lies inside an open disc in U

الحافة على الأقل

It follows that the distance from $z \in U$ to $\mathbb{C} \setminus U$ is at least $r_z > 0$, so that no point of U is a frontier point of U.

Examples:

(i) An open disc $B(z, r)$ and the half-plane $H = \{z : \mathrm{Re}\,(z) > 0\}$ are open sets.

(ii) If you take $L = \{z \in \mathbb{C} : |z| < 1$ and $\mathrm{Im}\, z \geq 0\}$ then $0 \in L$. But any disc $B(0, r)$ with $r > 0$ contains points not in L (e.g. $-ir/2$). So L is not open. Notice that 0 is a frontier point of L.

Openness and the frontier

One way to decide if a set $U \subseteq \mathbb{C}$ is open is to look at its frontier ∂U. This can be defined as the set of all $w \in \mathbb{C}$ such that there are points in U arbitrarily close to w AND points in $\mathbb{C} \setminus U$ arbitrarily close to w (in both cases these points are allowed to equal w).

For example, the sets $A = \{z \in \mathbb{C} : |z| < 1\}$, $B = \{z \in \mathbb{C} : |z| > 1\}$ and $C = \{z \in \mathbb{C} : |z| \leq 1\}$ all have the same frontier, namely the circle $T = \{z \in \mathbb{C} : |z| = 1\}$. Notice that $A \cap T = B \cap T = \emptyset$, whereas $C \cap T = T \neq \emptyset$.

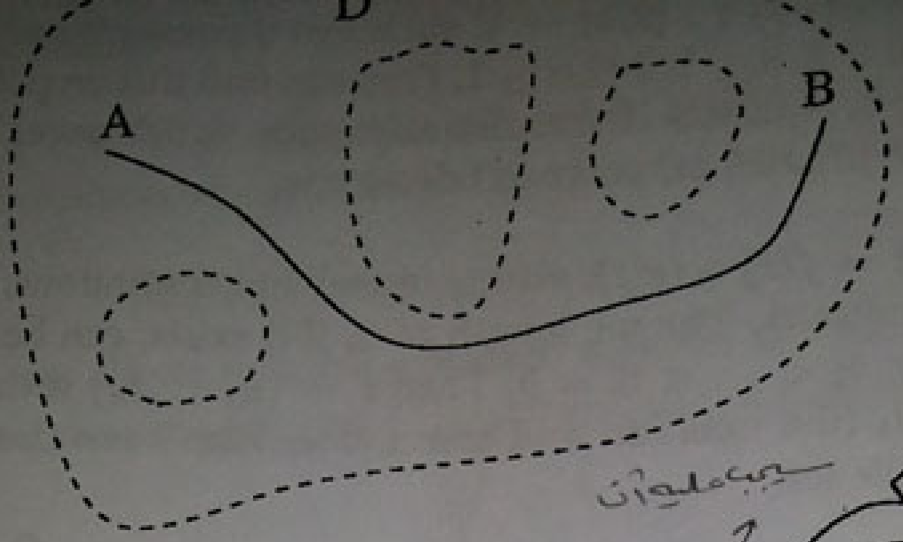
In fact, $U \subseteq \mathbb{C}$ is open if and only if $U \cap \partial U = \emptyset$. To see this, suppose first that U is open, and take w in U. Then $B(w, r) \subseteq U$ for some $r > 0$, so the distance from w to any point in $\mathbb{C} \setminus U$ is at least r, and so $w \notin \partial U$. Thus $U \cap \partial U = \emptyset$.

Now suppose that $U \cap \partial U = \emptyset$ and again take $w \in U$. Then $w \notin \partial U$ and so, since $w \in U$, there cannot be points of $\mathbb{C} \setminus U$ arbitrarily close to w. So there must exist $r > 0$ such that $|z - w| \geq r$ for all z in $\mathbb{C} \setminus U$, which means that $B(w, r) \subseteq U$ and so U is open.

So A and B above are open, but C is not. For the sets in G12COF it will usually be easy to determine the frontier, and so this criterion will be useful.

Domains

A domain is an open subset D of $\mathbb{C}$ which has the following additional property: any



An open disc is a domain, as is the half-plane $\mathrm{Re}\,(z) > 0$, but the set $\{z \in \mathbb{C} : \mathrm{Re}\,(z) \neq 0\}$ is not a domain, as any path $\gamma(t)$ from -1 to 1 would have to pass through $\mathrm{Re}\,(z) = 0$. This is using the intermediate value theorem and the fact that $\mathrm{Re}\,(\gamma(t))$ is continuous.

نظرية القيمة الوسطى نظرية مستمر

We will say that a set E in $\mathbb{R}^2$ is open (a domain) if the set in $\mathbb{C}$ corresponding to E, i.e. the set $\{x + iy : (x, y) \in E\}$, is open (a domain.)

يتوافق (التوافق)

2.9 Characterising constant functions on domains

Let D be a domain in $\mathbb{R}^2$, and let u be a real-valued function such that $u_x \equiv 0$ and $u_y \equiv 0$ on D. Then u is constant on D.

مشتق جزئي

Here the partial derivatives are defined by

$$u_x(a,b) = \frac{\partial u}{\partial x}(a,b) = \lim_{x \to a} \frac{u(x,b) - u(a,b)}{x - a}, \quad u_y(a,b) = \frac{\partial u}{\partial y}(a,b) = \lim_{y \to b} \frac{u(a,y) - u(a,b)}{y - b}.$$

Why is this fact true? Take any smooth contour $\gamma(t) = g(t) + ih(t) : [c,d] \to D$, where g, h are real. Then for $c < t < d$ we have, by the chain rule involving partial derivatives (G11CAL),

قاعدة السلسلة المشتقات الجزئية

$$\frac{d}{dt} u(\gamma(t)) = u_x g'(t) + u_y h'(t) = 0.$$

Hence u is constant on γ. Since any two points A, B in D can be linked by a PSC in D, and hence by finitely many smooth contours γ_k in D, joined end to end, we have $u(A) = u(B)$ and u is constant on D.

2.10 More remarks on paths and arc length (OPTIONAL!)

We have identified smooth contours as paths with nice properties suitable for subsequent application. However, paths in general are not always such simple objects

$$(t_n-t_0)(c-\delta)=\sum_{k=1}^{n}(t_k-t_{k-1})(c-\delta)<L(P)<\sum_{k=1}^{n}(t_k-t_{k-1})(c+\delta)=(t_n-t_0)(c+\delta).$$

Since P is an arbitrary partition of $[w,w+h]$ and $t_n-t_0=h$ we get

$$h(c-\delta)\le\Lambda(\gamma,w,w+h)=S(w+h)-S(w)\le h(c+\delta)$$

and

$$c-\delta\le\frac{S(w+h)-S(w)}{h}\le c+\delta. \tag{5}$$

Now let $0<|t|<\rho$. If $t>0$ then applying (5) with $w=v, h=t$ we obtain

$$\frac{S(v+t)-S(v)}{t}=\frac{S(w+h)-S(w)}{h}\in[c-\delta,c+\delta].$$

On the other hand if $t<0$ then we apply (5) with $h=-t, w=v+t, w+h=v$, to get

$$\frac{S(v+t)-S(v)}{t}=\frac{S(w)-S(w+h)}{-h}=\frac{S(w+h)-S(w)}{h}\in[c-\delta,c+\delta].$$

This says precisely that if $t\neq 0$ is small enough then

$$\frac{S(v+t)-S(v)}{t}$$

is as close as we need to c. Hence $S'(v)=c=|\gamma'(v)|$.

الدوال و الحدود

3 Functions and limits

الحدود

3.1 Limits

متتالية

We will define limits of functions using sequences.

غير منتهية قائمة

If (z_n) is a sequence (i.e. a non-terminating list) of complex numbers, we say that $z_n\to a\in\mathbb{C}$ as $n\to\infty$ if $|z_n-a|\to 0$ as $n\to\infty$ (i.e. the distance from z_n to a tends to 0).

For example, the sequence $z_n=e^{i/n}$ converges to 1 as $n\to\infty$.

يتقارب

مفرد - لا نظير له - وحيد

3.2 Functions on subsets of $\mathbb{C}$

If $E \subseteq \mathbb{C}$ a function f from E to $\mathbb{C}$ is a rule which assigns to each $z \in E$ a unique value $f(z) \in \mathbb{C}$. Such functions can be expressed either in terms of $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ or in terms of z and $\bar{z}$.

يمكن التعبير عنها

For example, consider $f(z) = \bar{z}z^2$.
If we put $x = \operatorname{Re}(z), y = \operatorname{Im}(z)$ then we have

$$f(z) = z(\bar{z}z) = z(x^2+y^2) = u(x,y) + iv(x,y)$$

where

$$u(x,y) = x(x^2+y^2), \quad v(x,y) = y(x^2+y^2).$$

من الصواب أن نكتب

It is standard to write

$$f(x+iy) = u(x,y) + iv(x,y), \tag{6}$$

with x, y real and u, v real-valued functions (of x and y).

معاكس

In the reverse direction we can write, for example,

$$(x+y)+iy = \frac{1}{2}(z+\bar{z}) + \frac{1}{2i}(z-\bar{z}) + i\frac{1}{2i}(z-\bar{z}) = z\left(1+\frac{1}{2i}\right) - \frac{1}{2i}\bar{z}$$

3.3 Limits

What do we mean by

$$\lim_{z\to a} f(z) = L \in \mathbb{C}?$$

تقترب من

We mean that as z approaches a, in any manner whatsoever, the value $f(z)$ approaches L. Here the value or existence of $f(a)$ makes no difference. Our definition is given in terms of sequences as follows.

لا تحدث فرق

قاعدة - مصطلح هام

Definition:
Let f be a complex-valued function defined near $a \in \mathbb{C}$ (but not necessarily at a itself). We say that $\lim_{z\to a} f(z) = L \in \mathbb{C}$ if the following is true. For every sequence (z_n) which converges to a with $z_n \neq a$, we have $\lim_{n\to\infty} f(z_n) = L$.

يتلاقى

This must hold for all sequences tending to, but not equal to, a, regardless of direction: the condition that $z_n \neq a$ is there because the existence or value of $f(a)$ makes no difference to the limit.

بغض النظر

Using the decomposition (6) (with x, y, u, v real) it is easy to see that

التحليل

$$\lim_{z\to a} f(z) = L \in \mathbb{C}$$

if and only if

$$\lim_{(x,y)\to(\operatorname{Re}(a),\operatorname{Im}(a))} u(x,y) = \operatorname{Re}(L) \quad \text{and} \quad \lim_{(x,y)\to(\operatorname{Re}(a),\operatorname{Im}(a))} v(x,y) = \operatorname{Im}(L).$$

This is because

$$|u - \mathrm{Re}\,(L)| + |v - \mathrm{Im}\,(L)| \le 2|f - L| \le 2(|u - \mathrm{Re}\,(L)| + |v - \mathrm{Im}\,(L)|).$$

A standard algebra of limits result is also true, proved in exactly the same way as in the real analysis case.

الجبر النهايات نتائج مثبتة تماما بنفس الطريقة

Example: Let

$$f(z) = \frac{|z|}{\pi + \mathrm{Arg}\,z}$$

for $z \neq 0$. Does $\lim_{z\to 0} f(z)$ exist?

Solution. If we let $z \to 0$ along some ray $\arg z = t$ with t in $(-\pi, pi)$, then the denominator is constant and $f(z) \to 0$. However, let $s > 0$ be small, and put $z = se^{i(-\pi+s^2)}$. Then $\mathrm{Arg}\,z = -\pi + s^2$ and so $f(z) = s/s^2 = 1/s \to \infty$ if we let s tend to 0 through positive values. So the limit doesn't exist.

استمرارية

3.4 Continuity

تعريف عام بالطريقة التقليدية

This is handled in the standard way. We say f is continuous at a if $\lim_{z\to a} f(z)$ exists and is $f(a)$. Thus $f(z)$ is as close as desired to $f(a)$, for all z sufficiently close to a.

بالقدر الذي نشاء

قريبة بالكفاية

Example:
At which points (if any) is Arg z discontinuous?

Solution.
Arg z is defined in $\mathbb{C} \setminus \{0\}$ and continuous everywhere, except for $z = x \in \mathbb{R}$ with $x < 0$. There $\mathrm{Arg}\,x = \pi$, but $\mathrm{Arg}\,z_n \to -\pi$ for $z_n = x - \frac{i}{n}$.

3.5 Complex differentiability

قابلية الاشتقاق

Let f be a complex-valued function defined on some open disc $B(a, r)$ and taking values in $\mathbb{C}$. We say that f is complex differentiable at a if there is a complex number $f'(a)$ such that

$$f'(a) = \lim_{z\to a} \frac{f(z) - f(a)}{z - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Examples:
1. Try $f(z) = \bar{z}$. Then we look at

$$\lim_{z\to a} \frac{\bar{z} - \bar{a}}{z - a} = \lim_{h\to 0} \frac{\bar{h}}{h}.$$

For $f'(a)$ to exist, the limit must be the same regardless of how h approaches 0. If we let $h \to 0$ through real values, we see that $\bar{h}/h = 1$. But, If we let $h \to 0$ through imaginary values, say $h = ik$ with k real, we see that $\bar{h}/h = -ik/ik = -1$. So $\bar{z}$ is not complex differentiable anywhere.

This is rather surprising, as $\bar{z}$ is a very well behaved function, and in everywhere continuous. If we write $\bar{z}$ in the form $u(x,y)+iv(x,y)$ we get $u=x$ and $v=-y$, and these have partial derivatives everywhere.

2. Try $f(z)=z^2$. Then, for any $a\in\mathbb{C}$,

$$\lim_{z\to a}\frac{z^2-a^2}{z-a}=\lim_{z\to a}(z+a)=2a,$$

by difference of two squares, and so the function z^2 *is* complex differentiable at every point, with $(d/dz)(z^2)=2z$ as expected.

In fact, the chain rule, product rule and quotient rules all apply just as in the real case. For example, the function

$$\frac{z^3-4}{z^2+1}$$

is complex differentiable at every point where $z^2+1\neq 0$, and so everywhere except i and $-i$.

المشتقة

3.6 The meaning of the complex derivative

In real analysis we think of $f'(x_0)$ as the slope of the graph of f at x_0. In complex analysis it doesn't make sense to attempt to "draw a graph" since both z and $f(z)$ live in two dimensions.

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However, we can think of the derivative in terms of approximation. If f is complex differentiable at a then as $z\to a$ we have

قابلة للتفاضل

$$\frac{f(z)-f(a)}{z-a}\to f'(a)$$

and so

$$\frac{f(z)-f(a)}{z-a}=f'(a)+\rho(z),$$

where $\rho(z)\to 0$ as $z\to a$. We can write this as

$$f(z)-f(a)=(z-a)(f'(a)+\rho(z)). \tag{7}$$

In particular, the right hand side of (7) tends to 0 as $z\to a$ and so f is continuous at a. Equation (7) says that as $z\to a$ the function $(z-a)f'(a)$ is a good approximation to $f(z)-f(a)$.

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We can use the formula (7) to check the chain rule. Suppose that g is complex differentiable at z_0 and f is complex differentiable at $w_0=g(z_0)$. As $z\to z_0$ we have

$$\frac{g(z)-g(z_0)}{z-z_0}\to g'(z_0),$$

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where $\rho(z)\to 0$ as $z\to z_0$. Similarly

$$f(w)=f(w_0)+(w-w_0)(f'(w_0)+\tau(w))$$

where $\tau(w)\to 0$ as $w\to w_0$. Substitute in $w=g(z), w_0=g(z_0)$. Then

$$\begin{aligned} f(g(z)) &= f(g(z_0))+(g(z)-g(z_0))(f'(g(z_0))+\tau(g(z))) \\ &= f(g(z_0))+(z-z_0)(g'(z_0)+\rho(z))(f'(g(z_0))+\tau(g(z))). \end{aligned}$$

As $z\to z_0$ we have $\rho(z)\to 0$ and $g(z)\to w_0$ so that $\tau(g(z))\to 0$. Hence

$$\frac{f(g(z))-f(g(z_0))}{z-z_0}=(g'(z_0)+\rho(z))(f'(g(z_0))+\tau(g(z)))\to g'(z_0)f'(g(z_0))$$

as $z\to z_0$, giving the expected formula $(f(g))'=f'(g)g'$.

كوشي ريمان علاقات مواجهة

3.7 Cauchy-Riemann equations, first encounter

Assume that the complex-valued function f is complex differentiable at $a=A+iB$, and as in (6) write

$$f(x+iy)=u(x,y)+iv(x,y)$$

with A,B,x,y,u,v all real. Now, by assumption, → الفرضيات

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)\in\mathbb{C}$$

بغض النظر

and the limit is the same regardless of how h approaches 0. So if we let h approach 0 through real values, putting $h=t$, we get

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$$\begin{aligned} f'(a) &= \lim_{t\to 0}\frac{f(a+t)-f(a)}{t} \\ &= \lim_{t\to 0}\frac{u(A+t,B)-u(A,B)+iv(A+t,B)-iv(A,B)}{t} \\ &= u_x(A,B)+iv_x(A,B). \end{aligned}$$

In particular, the partial derivatives u_x, v_x do exist. Next, put $h=it$ and again let $t\to 0$ through real values. This time we get

$$f'(a)=\lim_{t\to 0}\frac{u(A,B+t)-u(A,B)+iv(A,B+t)-iv(A,B)}{it}=\frac{1}{i}(u_y(A,B)+iv_y(A,B)).$$

Equating real and imaginary parts we now see that, at the point (A,B), we must have

[illegible] $$\boxed{u_x=v_y\ ,\quad u_y=-v_x.}$$

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بالأهم

These are called the Cauchy-Riemann equations. We also have (importantly) $f'(a) = u_x + iv_x$. These relations must hold if f is complex differentiable at the point $a = A + iB$.

يجب أن [illegible]

Consider $f(z) = \bar{z}$, for which $u = x$ and $v = -y$. We have

$$u_x = 1, \quad v_y = -1,$$

محققة منها

and so the Cauchy-Riemann equations are never satisfied, from which it follows that $\bar{z}$ is nowhere complex differentiable.

This has given us a *necessary* condition for a function $f(z)$ to be complex differentiable. Next we need a result in the other direction, i.e. a *sufficient* condition.

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3.8 Cauchy-Riemann equations, second encounter

Are the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x,$$

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sufficient to ensure that the function $f = u + iv$ is complex differentiable at some point?

If we define

$$f(x+iy) = 0 \quad (\text{if } x = 0 \text{ or } y = 0) \quad f(x+iy) = 1 \quad (\text{otherwise})$$

محور تخيلي

then on the real and imaginary axes we have $u = v = 0$ and so at $(0,0)$ we have $u_x = v_y = u_y = v_x = 0$. So the Cauchy-Riemann equations are satisfied at this point. But $f(x+ix) = 1$ for $x > 0$, so that $f(x+ix)$ does not tend to $f(0) = 0$ as $x \to 0$. Hence f is not continuous at 0, and so is definitely not complex differentiable at 0.

It turns out that in order to conclude that a function is complex differentiable it suffices to assume in addition to the Cauchy-Riemann equations that the partial derivatives are continuous.

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Theorem

Suppose that the functions f, u, v are as in (6) above, and that the following is true. The partial derivatives u_x, u_y, v_x, v_y all exist near (A, B), and are continuous at (A, B), and the Cauchy-Riemann equations are satisfied at (A, B). Then f is complex differentiable at $a = A + iB$, with $f'(a) = u_x + iv_x$.

نفرض أن

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Remark: the continuity of the partial derivatives will not usually be a problem in G12COF, e.g. this is automatic if they are polynomials in x, y and (say) functions like $e^x, \cos y$. If there are denominators which vanish at (A, B) some care is needed, though.

Example:

At which points, if any, is $x^2 + iy^2$ complex differentiable?

Solution. We set $u(x,y) = x^2$ and $v(x,y) = y^2$. Then $u_x = 2x$, $u_y = v_x = 0$ and $v_y = 2y$.

تحليلي

ع الترتيب

تعريف

3.9 Analytic functions

هام

We say that f is ANALYTIC at a point a (respectively analytic on a set X) if f is complex differentiable on an open set G which contains the point a (respectively the set X). Obviously, if f is complex differentiable on a domain D in $\mathbb{C}$ then f is analytic on D (take $G = D$). Other words for analytic are regular, holomorphic and (in very old books) uniform. A sufficient condition for analyticity at a is that the partial derivatives of u, v are continuous and satisfy the Cauchy-Riemann equations at all points near a.

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3.10 Examples of analytic functions

1. The exponential function. We have already defined $e^{it} = \cos t + i \sin t$ for t real. We now define

$$\exp(x+iy) = e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y$$

for x, y real. We then have, using the standard decomposition,

$$u(x,y) = e^x \cos y \quad , \quad v(x,y) = e^x \sin y ,$$

and

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = e^x \sin y, \quad v_y = e^x \cos y.$$

Hence the Cauchy-Riemann equations are satisfied at every point, and obviously these partial derivatives are continuous. Thus $\exp(z)$ is complex differentiable at every point in $\mathbb{C}$, and so is analytic in $\mathbb{C}$, or ENTIRE. Further, the derivative of exp at z is $u_x + i v_x = \exp(z)$.

It is easy to check that $e^{z+w} = e^z e^w$ for all complex z, w. Indeed, suppose that

$$z = x+iy, \quad w = u+iv, \quad x,y,u,v \in \mathbb{R}.$$

Then

$$e^{z+w} = e^{x+u+i(y+v)} = e^{x+u} e^{i(y+v)} = e^x e^u e^{iy} e^{iv},$$

which is what we need. Here the fact that

$$e^{i(y+v)} = e^{iy} e^{iv}$$

comes from the trig formulas

$$\cos(y+v) = \cos y \cos v - \sin y \sin v, \quad \sin(y+v) = \sin y \cos v + \cos y \sin v.$$

Also $|e^z| = e^{\mathrm{Re}(z)} \neq 0$, so $\exp(z)$ never takes the value zero. Since $e^0 = e^{2\pi i} = 1$ and $e^{\pi i} = -1$ this means that two famous theorems from real analysis are not true for functions of a complex variable (if you want to know which, see Section 9.3)!

2. sine and cosine. For $z \in \mathbb{C}$ we set

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad , \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} .$$