تكاملات التوابع الكسرية:

* نقول عن التابع أنه كسري إذا كان من الشكل $\frac{F(x)}{G(x)}$ حيث $F(x)$ كثير حدود و $G(x)$ كثير حدود

① درجة المقام $F(x)$ > درجة المقام $G(x)$

نقسم البسط على المقام فنجد: $\frac{F(x)}{G(x)} = T(x) + \frac{F(x)}{G(x)}$ ← بقسمة

طريقة تفريق العوامل: $G(x) = (x-a_1)^k (x-a_2)^r \dots (x-a_m)^e$ (P)

يوجد في دفتر على (1) صفحة

$$\frac{F(x)}{G(x)} = \frac{A_1}{(x-a_1)^k} + \frac{A_2}{(x-a_1)^{k-1}} + \dots + \frac{A_k}{(x-a_1)} + \frac{B_1}{(x-a_2)^r} + \frac{B_2}{(x-a_2)^{r-1}} + \dots$$

$$+ \frac{B_r}{(x-a_2)} + \dots + \frac{C_1}{(x-a_m)^e} + \dots + \frac{C_e}{(x-a_m)}$$

$$G(x) = (a_1x^2 + b_1x + c_1)^k \dots (a_mx^2 + b_mx + c_m)^r \quad (U)$$

$$\frac{F(x)}{G(x)} = \frac{A_1x + B_1}{(a_1x^2+b_1x+c_1)^k} + \frac{A_1x + B_2}{(a_1x^2+b_1x+c_1)^{k-1}} + \dots + \frac{A_kx + B_k}{(a_1x^2+b_1x+c_1)}$$

$$+ \dots + \frac{E_1x + D_1}{(a_mx^2 + b_mx + c_m)^r} + \dots$$

مثال:

① $$I = \int \frac{x}{x^3 - 3x + 2}\, dx$$

$$(x^3 - 3x + 2) = (x-1)(x^2 + x - 2)$$

$$= (x-1)(x-1)(x+2) = (x-1)^2(x+2)$$

SABBAGH STATIONERS

⑤ التكامل بالتجزئة نفرق مقوا

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$x = (A+C)x^2 + (A+B-2C)x + (-2A+2B+C)$$ بالتطابق

x^2	$0 = A + C \Rightarrow A = -C$
x	$1 = A + B - 2C \Rightarrow 1 + 3C = B$
الثوابت	$0 = -2A + 2B + C$

$$C = -\frac{2}{9}, \quad A = \frac{2}{9}, \quad B = \frac{1}{3}$$

$$I = \int \frac{x}{(x-1)^2(x+2)}\,dx = \frac{2}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2} - \frac{2}{9}\int \frac{dx}{x+2}$$

$$= \frac{2}{9}\ln|x-1| + \frac{1}{3}\,\frac{(x-1)^{-1}}{-1} - \frac{2}{9}\ln|x+2| + c$$

$$= \frac{2}{9}\ln\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c$$

(2) $$J = \int \frac{x^2}{(x-1)(x^2+1)}\,dx$$

$$\frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$x^2 = A(x^2+1) + (Bx+C)(x-1)$$

$$= (A+B)x^2 + (C-B)x + (A-C)$$

SARBAGH STATIONERY

$1 = A + B \qquad 1 = C + C \Rightarrow \frac{1}{2} = C = B = C$

$0 = C - B \Rightarrow C = B$

$0 = A - C \qquad A = C$

$$J = \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{x+1}{x^2+1}\,dx$$

تفرق

$$= \frac{1}{2}\ln|x-1| + \frac{1}{2}\int \frac{x\,dx}{x^2+1} + \frac{1}{2}\int \frac{1}{x^2+1}$$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{4}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\operatorname{arctg} x$$

$$= \frac{1}{2}\operatorname{Ln}|x-1| + \frac{1}{4}|x^2+1| + \frac{1}{2}\operatorname{arctg} x + C$$

المحاضرة الخامسة ٤/٣/ ٢٠١٠ م.

تكاملات تؤدي إلى تكاملات كسرية:

$$I = \int R(x, x^{r_1}, x^{r_2}, \dots x^{r_n})\,dx \qquad ; r_1, r_2 \dots \in Q$$

$x = t^q$ حيث q مضاعف المشترك الأصغر لمقامات الأعداد $r_1, r_2 \dots r_n$

① $I = \int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})}$ $\qquad r_1 = \frac{1}{2} \qquad r_2 = \frac{1}{3} \qquad q = 6$ المضاعف المشترك

$$x = t^6 \Rightarrow dx = 6t^5\,dt$$

$$I = 6\int \frac{t^5\,dt}{t^6(1+2t^3+t^2)} = 6\int \frac{t^5\,dt}{t^6(1+2t^3+t^2)} = 6\int \frac{dt}{t(1+2t^3+t^2)}$$

SARBAGH STATIONERY

$$\begin{array}{r|l} 2t^2 - t + 1 & \leftarrow \text{إذن } t = -1 \\ t+1 \;\overline{)\; 2t^3 + t^2 + 1} & \text{أي } t+1 = 0 \\ 2t^3 + 2t^2 & \\ \hline -t^2 + 1 & \\ -t^2 - t & \\ \hline t + 1 & \\ t + 1 & \\ \hline \end{array}$$

$$= 6 \int \frac{dt}{t(t+1)(2t^2 - t + 1)}$$

$$\frac{1}{t(t+1)(2t^2-t+1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{ct+D}{2t^2-t+1}$$

$$1 = A(2t^3 + t^2 + 1) + B(2t^3 - t^2 + t) + (ct + D)(t^2 + t)$$

$$0 = 2A + 2B + C$$

$$0 = A + B + C + D$$

$$0 = 0 + B + D$$

$$1 = A + 0 \qquad B = -\frac{1}{4} \qquad D = \frac{1}{4}$$

$$\Rightarrow C = -\frac{3}{2} = -\frac{6}{4}$$

$$\frac{1}{t(t+1)(2t^2-t+1)} = \frac{1}{t} + \frac{1}{4(t+1)} - \frac{6t-1}{4(2t-t+1)}$$

SARRAGH STATIONERY

$$I = 6\int \frac{dt}{t} - \frac{6}{4}\int \frac{dt}{t+1} - \frac{6}{4}\int \frac{6t-1}{(2t^2-t+1)}$$

$$= 6\ln|t| - \frac{6}{4}\ln|t+1| - \frac{6}{4}\int \frac{(4t-1)+2t}{(2t^2-t+1)}\,dt$$

$$I = \ln\left|\frac{t^6}{(t+1)^{\frac{6}{4}}}\right| - \frac{6}{4}\int \frac{(4t-1)\,dt}{2t^2-t+1} - 3\int \frac{t\,dt}{2t^2-t+1}$$

$$= \ln\left|\frac{t^6}{(t+1)^{\frac{6}{4}}}\right| - \frac{6}{4}\ln(2t^2-t+1) - 3\,(I_1) \longrightarrow \text{سوف نحل}$$

$$I_1 = \int \frac{t\,dt}{2t^2-t+1} = \frac{1}{4}\int \frac{4t+1-1}{2t^2-t+1}\,dt$$

$$I_1 = \frac{1}{4}\int \frac{4t-1}{2t^2-t+1}\,dt + \frac{1}{4}\int \frac{dt}{2t^2-t+1}$$

$$= \frac{1}{4}\int \frac{4t-1}{2t^2-t+1}\,dt + \frac{1}{4}\int \underbrace{\frac{dt}{2t^2-t+1}}_{\text{نكمل المربع}} \qquad \frac{dt}{2\left(t^2-\frac{t}{2}+\frac{1}{16}-\frac{1}{16}+\frac{1}{2}\right)}$$

$$= \frac{1}{4}\ln|2t^2-t+1| + \frac{1}{8}\int \frac{dt}{\left(t+\frac{1}{4}\right)^2+\frac{7}{16}} \qquad t-\frac{1}{4} = u$$

$$dt = du$$

$$\left(\frac{I}{1}\right) = \frac{1}{4}\ln|2t^2-t+1) + \frac{4}{8\sqrt{7}}\,\text{arctg}\left(t-\frac{1}{4}\right)\frac{4}{\sqrt{7}} + C \qquad \int \frac{du}{u^2+a^2}$$

لنعوضه بحل (I) وكتبه الحصار

SARBAGH STATIONERY

② $I = \int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{r_1}, \left(\frac{ax+b}{cx+d}\right)^{r_2}, \dots\right) dx$

دالة ناطقة

$$t^q = \frac{ax+b}{cx+d}$$

q المضاعف المشترك لمقامات الأعداد $r_n \dots r_2, r_1$

① $= \int \frac{\sqrt{2x-3}}{\sqrt[3]{2x-3}+1}\, dx = \int \frac{(2x-3)^{\frac{1}{2}}}{(2x-3)^{\frac{1}{3}}+1}$ $\quad r_1 = \frac{1}{2}$, $r_2 = \frac{1}{3}$, $q = 6$

نفرض

$2x-3 = t^6 \Rightarrow 2dx = 6t^5\, dt$

$\Rightarrow dx = 3t^5\, dt$

$$= \int \frac{t^3}{t^2+1} 3t^5\, dt = 3\int \frac{t^8}{t^2+1}$$

$$= 3\int (t^6 - t^4 + t^2 - 1)\, dt + 3\int \frac{dt}{t^2+1}$$

$$= 3\left(\frac{t^7}{7} - \frac{t^5}{5} + \frac{t^3}{3} - t\right) + 3\arctan t + c$$

ونعوض: $t = (2x-3)^{\frac{1}{6}}$

$t^8 \div (t^2+1)$:

$t^8 + t^6$

$-t^6$

$-t^6 - t^4$

t^4

$t^4 + t^2$

$-t^2$

$-t^2 - 1$

$+1$

$t^6 - t^4 + t^2 - 1$

© ABNAGH STATIONERY

(1) $= \int \frac{dx}{\cos^3 x} =$ حل تمرين من الوضعية :

$= \int \frac{dx}{\cos x \cdot \cos^2 x}$ $\quad u = \frac{1}{\cos x} \Rightarrow du = \frac{\sin x}{\cos x}$

$dv = \frac{dx}{\cos^2 x} \Rightarrow v = tg\,x$

$I = \frac{\sin x}{\cos^2 x} - \int \frac{\sin^2 x}{\cos^2 x}\, dx$ $\quad I_1$

$\sin x = t \Rightarrow \cos x\, dx = dt \Rightarrow dx = \frac{dt}{\cos x} = \frac{dt}{\sqrt{1+t^2}}$

$= \int \frac{t^2\, dt}{\sqrt{1-t^2}\sqrt{1-t^2}(1-t^2)} = \int \frac{t^2}{(1-t^2)^2} = \int \frac{t^2}{(1-t)^2(1+t)^2}$

نستعمل طريقة المعاملات المجهولة A , B , C , D

$A \int \frac{dt}{1+t} + B \int \frac{dt}{(1-t)^2} + C \int \frac{dt}{1+t} + d \int \frac{dt}{(1+t)^2}$

$\ln$

$A = -\frac{1}{2} \quad B = \frac{1}{2} \quad C = -\frac{1}{2} \quad D = \frac{1}{2}$

$I_1 = \frac{1}{2}\ln(1-t) + \frac{1}{2}\frac{1}{(1-t)} - \frac{1}{2}\ln(1+t) - \frac{1}{2(t+1)}$

$-t+1 = u$
$-dt = du$
$\int \frac{t\, du}{u^2}$

$I_1 = \frac{1}{2}\ln\left|\frac{1-\sin x}{1+\sin}\right| + \frac{1}{2(1-\sin x)} - \frac{1}{2(1+\sin x)}$

SABBAGH STATIONERY

$$I = \int \frac{1}{\cos x} tg^2 x \, dx$$

$$= \int \frac{\sin^2 x}{\cos^3 x} \, dx$$

المحاضرة السادسة ١١/٣/٢٠١٠ م.

تحويلات أويلر الأولى:

تستخدم إذا كانت التكاملات من الشكل:

$$I = \int R\left(x, \sqrt{ax^2+bx+c}\right) dx$$

① تحويل أويلر الأول: $a>0$

$$\sqrt{ax^2+bx+c} = \mp\sqrt{a}\, x \mp t$$

نربع الطرفين:

$$ax^2+bx+c = ax^2 + 2\sqrt{a}\, xt + t^2$$

$$x(b - 2\sqrt{a}\, t) = t^2 - c$$

$$x = \frac{t^2-c}{b-2\sqrt{a}\, t} \Rightarrow \quad dx = ?$$

مثال:

$$I = \int \frac{(x+2)}{(x-1)\sqrt{x^2+2x+5}}$$

$$= \sqrt{x^2+2x+5} = x + t$$

$$x^2+2x+5 = x^2 + 2xt + t^2$$

$$x(2-2t) = t^2 - 5 \Rightarrow \frac{t^2-5}{2(1-t)}$$

SABBAGH STATIONERY

$$dx = \frac{4t(1-t) + 2(t^2-5)}{4(1-t)^2}$$

$$= \frac{-2t^2 + 4t - 10}{4(1-t)^2}\,dt$$

$$dx = \frac{-t^2 + 2t - 5}{2(1-t)^2}\,dt$$

$$\sqrt{x^2 + 2x - 5} = x + t$$

$$\Rightarrow \frac{t^2 - 5}{2(1-t)} - \frac{t^2 - 5 + 2t - 2t^2}{2(1-t)}$$

$$\sqrt{x^2 + 2x - 5} = \frac{-t^2 + 2t - 5}{2(1-t)}$$

$$x + 2 = \frac{t^2 - 5}{2(1-t)} + 2 = \frac{t^2 - 5 + 4 - 4t}{2(1-t)}$$

$$x + 2 = \frac{t^2 - 4t - 1}{2(1-t)}$$

$$x - 1 = \frac{t^2 - 2}{2(1-t)} - 1 = \frac{t^2 - 5 - 2 + 2t}{2(1-t)}$$

$$x - 1 = \frac{t^2 - 2t - 7}{2(1-t)}$$

$$I = \int \frac{\frac{t^2 - 4t - 1}{2(1-t)} \left(\frac{-t^2 + 2t - 5}{2(1-t)}\right)}{\frac{(t^2 + 2t - 7)(-t^2 + 2t - 5)}{2(1-t)\ 2(1-t)}}$$

$$= \int \frac{t^2 - 4t - 1}{(1-t)(t^2 + 2t - 7)}\,dt$$

$$dx = \frac{-4(t^2+1) - 2t(3-4t)}{(t^2+1)^2}\,dt$$

$$dx = \frac{4t^2 - 6t - 4}{(t^2+1)^2}\,dt$$

$$x - 1 = \frac{3-4t}{t^2+1} - 1 = \frac{3-4t-t^2-1}{t^2+1}$$

$$x - 1 = \frac{-t^2-4t+2}{t^2+1}$$

$$\sqrt{-x^2+3x+4} = 2 + \frac{3-4t}{t^2+1} \cdot t$$

$$\sqrt{\quad} = \frac{2t^2+2+3t-4t^2}{t^2+1}$$

$$\sqrt{\quad} = \frac{-2t^2+3t+2}{t^2+1}$$

$$I = \int \frac{\frac{-t^2-4t+2}{t^2+1} \cdot \frac{4t^2-6t-4}{(t^2+1)^2}}{\frac{3-4t}{t^2+1}\left(\frac{-2t^2+3t+2}{t^2+1}\right)}\,dt$$

$$= -2\int \frac{-t^2-4t+2}{(t^2+1)(3-4t)}\,dt$$

$$= -2\int \frac{t^2+4t-2}{(t^2+1)(3-4t)}\,dt$$

SARRAH STATIONERY