$$= 2\int \frac{A}{3-4t}\,dt + 2\int \frac{Bt+c}{t^2+1}\,dt$$

$$A=1 \quad , B=0 \quad , c=-1$$

$$I = 2\int \frac{dt}{3-4t} - 2\int \frac{dt}{t^2+1}$$

$$= -\frac{1}{2}\ln|3-4t| - 2\,\text{cotg}\,t + c$$

$$t = \frac{\sqrt{\quad} - 2}{x}$$

تحويل أولر الثالث :

إذا كان . $ax^2 + bx + c = a(x-\alpha)(x-\beta)$

إما $\sqrt{ax^2+bx+c} = (x-\alpha)t$

أو $\cancel{\sqrt{ax^2+bx+c}}$

$$\sqrt{a(x-\alpha)(x-\beta)} = (x-\beta)^2 t^2$$

$$a\left(\frac{x-\alpha}{x-\beta}\right) = t^2 \Rightarrow x = \varphi(t)$$

$$\Rightarrow dx =$$

تمرين :

$$I = \int \frac{x\,dx}{\sqrt{x^2+5x+6}}$$

$$x^2+5x+6 = (x+2)(x+3)$$

SABBAGH STATIONERY

$$\sqrt{(x+2)(x+3)} = (x+3)\,t$$

$$(x+2)(x+3) = (x+3)^2\,t^2$$

$$x+2 = (x+3)\,t^2$$

$$x(t^2-1) = 2-3t^2$$

$$x = \frac{2-3t^2}{t^2-1}$$ نعزل x ومن ثم نشتق

$$dx = \frac{-6t(t^2-1)-2t(2-3t^2)}{(t^2-1)^2}$$

$$\boxed{dx = \frac{+2t\,dt}{(t^2-1)^2}}$$

$$\sqrt{x^2+5x+6} = \left(\frac{2-3t^2}{t^2-1} - 3\right)t$$

$$= \left(\frac{2-\cancel{3t^2}+\cancel{3t^2}-3}{t^2-1}\right)t$$

$$\boxed{\sqrt{} = \frac{-t}{t^2-1}}$$

$$I = \int \frac{\left(\frac{2-3t^2}{t^2-1}\right)\left(\frac{2t}{(t^2-1)^2}\right)}{\frac{-t}{t^2-1}}\,dt$$

$$= -2\int \frac{2-3t^2}{(t^2-1)^2}\,dt$$

SARRAGH STATIONERY

$$= 2\int \frac{3t^2-2}{(t-1)^2(t+1)^2}\,dt = 2A\int\frac{dt}{(t-1)} + 2B\int\frac{dt}{t+1} + 2D\int\frac{dt}{(t+1)^2}$$

$$+ 2C\int\frac{dt}{t+1}$$

$$= 2A\ln|(t-1)| + 2C\ln|t+1| - 2\frac{B}{t-1} - 2\frac{D}{t+1} + C$$

طرق خاصة لحساب:

$$I = \int \frac{Q(x)}{\sqrt{ax^2+bx+c}}\,dx$$

$Q(x)$ كثير حدود.

$$\int \frac{Q(x)\,dx}{\sqrt{ax^2+bx+c}} = F(x)\cdot\sqrt{ax^2+bx+c} + \lambda\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

كثير حدود درجته أقل من درجة $Q(x)$ بواحد

لتعيين الثوابت λ وأمثال كثير الحدود $F(x)$

نشتق

$$\frac{Q(x)}{\sqrt{}} = F'(x)\sqrt{} + \frac{(2ax+b)F(x)}{2\sqrt{}} + \frac{\lambda}{\sqrt{}}$$

$$Q(x) = F'(x)(ax^2+bx+c) + \frac{(2ax+b)F(x)}{2} + \lambda$$

نطابق بين الأمثال لإيجاد ثوابت $F(x)$ و λ .

$$I = \int \frac{6x^3 - x - 4}{\sqrt{x^2-2x-3}}\, dx = (ax^2+bx+c)\left(\sqrt{x^2-2x-3}\right) + h\int \frac{dx}{\sqrt{x^2-2x-3}}$$

$$= \left[(2ax+b)\sqrt{x^2-2x-3}\right] + (ax^2+bx+c)\frac{2x-2}{2\sqrt{x^2-2x-3}} + \frac{h}{\sqrt{x^2-2x^3}}$$

$$6x^3 - x - 4 = (2ax+b)(x^2-2x-3) + (x-1)(ax^2+bx+c) + h$$

$$6 = 2a + a \Rightarrow a = 2$$

$$0 = b - 4a - a + b \Rightarrow 2b = 10 \Rightarrow b = 5$$

$$-1 = -6a - 2b + c - b \Rightarrow c = 26$$

$$-4 = -3b - c + h \Rightarrow h = 37$$

$$I = \int (2x^2 + 5x + 26)\left(\sqrt{x^2-2x-3}\right) + 37\underbrace{\int \frac{dt}{\sqrt{x^2-2x-3}}}_{K}$$

$$K = \int \frac{dt}{\left(\sqrt{x^2-2x+1-1-3}\right)} = \int \frac{dt}{(x-1)^2-(2)^2} = \text{arc}$$ تکمیل شود.

SABBAGH STATIONERY

المحاضرة السابعة ١٤/٣/٢٠١٠

التكاملات المثلثية:

$$\int f(\sin x \cos x)\,dx$$

$$tg\frac{x}{2} = t \Rightarrow \frac{1}{2}\left(1+tg^2\frac{x}{2}\right)dx = dt \Rightarrow dx = \frac{2dt}{1+t^2}$$

$$\frac{1}{\cos^2\frac{x}{2}} = 1+tg^2\frac{x}{2} = 1+t^2 \qquad \left(\cos x = 2\cos^2\frac{x}{2} - 1\right)$$

$$\frac{2}{\cos x+1} = 1+t^2 \Rightarrow \frac{\cos x+1}{2} = \frac{1}{1+t^2}$$

$$\Rightarrow \cos x = \frac{2}{1+t^2} - 1 = \frac{2-1-t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \sqrt{1-\left(\frac{1-t^2}{1+t^2}\right)^2} = \frac{\sqrt{1+2t^2+t^4+2t^2-1-t^4}}{1+t^2}$$

$$\sin x = \frac{\sqrt{4t^2}}{1+t^2} = \frac{2t}{1+t^2}$$

$$I = \int f(\sin x)\cos x\,dx$$

$$\sin x = t \Rightarrow \cos x\,dx = dt$$

$$I = \int f(t)\,dt$$

$$J = \int f(\cos x)\sin x\,dx$$

$$\cos x = t \Rightarrow -\sin x\,dx = dt$$

$$J = -\int f(t)\,dt$$

$$K = \int f(tg x)\,dx$$

SABBAGH STATIONERY

$$tgx = t \Rightarrow (1 + tg^2 x)\,dx = dt \Rightarrow dx = \frac{dt}{1 + tg^2 x}$$

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{1}{1+t^2}$$

$$\sin^2 x = \frac{t^2}{1+t^2}$$

$$\cos^2 x = \frac{1}{1+tg^2 x} = \frac{1}{1+t^2} \Rightarrow \cos^2 x = \frac{1}{1+t^2}$$

$$I = \int f(\cos x \sin x)\,dx$$

حيث تكون $\sin x$ و $\cos x$ زوجية نستخدم التحويل $tgx = t$

$$I = \int \sin^m x \cos^n x \, dx \qquad m, n \in \mathbb{Z}$$

① m فردي و n زوجي

$m = 2p+1$ ، $n = 2q$

$$= \int \sin^{2p+1} x \cdot \cos^{2p} x \, dx$$

$$= \int \sin x^{2p} \cos x^{2p} \sin x \, dx$$

$$\cos x = t \Rightarrow -\sin x\,dx = dt \quad ، \quad \sin^2 x = 1 - \cos^2 x = 1 - t^2$$

② m زوجي و n فردي

$n = 2p+1$ $\qquad$ $m = 2p$

$$= \int \sin x^{2q} \cdot \cos x^{2p+1} \, dx$$

$$= \int \sin x^{2q} \cos x^{2p} \cos x \, dx$$

$$\sin x = t \Rightarrow \cos x\,dx = dt \quad , \quad \cos^2 x = 1 - \sin^2 x = 1 - t^2$$

$$\cancel{I = \int (1-t^2)^p \, t^q \, dt}$$

$$I = \int t^{2q} (1-t^2)^p \, dt$$

③ m زوجي ، n فردي

$n = 2p\cancel{+1}$ $\quad m = 2q$

$$= \int \sin^{2q} \cos x^{2p+1} \, dx$$

$$= \int \sin x^{2q} \cos x^{2p} \cdot \cos x \, dx$$

$\sin x = t \Rightarrow \cos x \, dx = dt \quad , \cos^2 x = 1 - \sin^2 x = 1 - t^2$

$$\cos^2 x = \frac{1 + \cos 2x}{2} \quad , \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\int \left(\frac{1 + \cos 2x}{2}\right)^q \left(\frac{1 - \cos 2x}{2}\right)^p dx$$

① $$I = \int \frac{dx}{\cos x}$$

$tg \frac{x}{2} = t \Rightarrow dx = \frac{2dt}{1+t^2} \quad , \cos x = \frac{1-t^2}{1+t^2}$

$$I = \int \frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{1-t^2} = 2\int \frac{dt}{(1-t)(1+t)} = \int \frac{dt}{1-t} + \int \frac{dt}{1+t}$$

$$= -\ln|1-t| + \ln|1+t| + c$$

$$I = \ln\left|\frac{1 + tg\frac{x}{2}}{1 - tg\frac{x}{2}}\right| + c$$

SABBAGH STATIONERY

$$* \quad I = \int \sin^2 x \cos^2 x \, dx$$

$$= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right) dx$$

$$= \frac{1}{4} \int 1 - \cos^2 2x \, dx$$

$$= \frac{x}{4} - \frac{1}{4} \int \cos^2 2x \, dx$$

$$= \frac{x}{4} - \frac{1}{4} \int \frac{1 + \cos 4x}{2} \, dx$$

$$= \frac{x}{4} - \frac{x}{8} - \frac{1}{8} \frac{\sin 4x}{4}$$

$$* \quad I = \int \frac{\sin^5 x}{\cos^2 x} \, dx = \int \frac{\sin^4 x}{\cos^2 x} \sin x \, dx$$

$$\cos x = t \Rightarrow -\sin x \, dx = dt$$

$$I = -\int \frac{(1-t^2)^2}{t^2} \, dt = -\int \frac{1 - 2t^2 + t^4}{t^2} \, dt$$

$$= -\int \frac{dt}{t^2} + 2 \int dt - \int t^2 \, dt$$

$$= \frac{1}{t} + 2t - \frac{t^3}{3} + c$$

$$= \frac{1}{\cos x} + 2 \cos x - \frac{\cos^3 x}{3} + c$$

SABBAGH STATIONERY

$$* \ I = \int \cos^4 x \sin^2 x \, dx$$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{8} \int (1+\cos 2x)(1-\cos^2 2x) \, dx$$

$$= \frac{1}{8} \int (1+\cos 2x - \cos^2 2x - \cos^3 2x) \, dx$$

$$= \frac{1}{8} \left[x + \frac{\sin 2x}{2} - \int \frac{1+\cos 4x}{2} dx - \int \cos^3 2x \, dx \right.$$

$$= \frac{1}{8} \left[x + \frac{\sin 2x}{2} - \frac{x}{2} - \frac{\sin 4x}{8} - \underbrace{\int \cos^2 2x \cos 2x \, dx}_{\mathfrak{z}} \right.$$

$$\mathfrak{z} = \int \cos^2 2x \cos 2x \, dx$$

$$\sin 2x = t \Rightarrow 2\cos 2x \, dx = dt$$

$$= \frac{1}{2} \int (1 - t^2) \, dt = \frac{t}{2} - \frac{t^3}{6}$$

$$= \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}$$

$$= \frac{1}{8} \left[x + \frac{\sin 2x}{2} - \frac{x}{2} - \frac{\sin 4x}{8} - \frac{\sin 2x}{2} + \frac{\sin^3 2x}{6} \right]$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{(\sin 2x)^3}{48} + c$$

SABBAGH STATIONERY

التكاملات المثلثية:

$$I = \int f(sh x, ch x)\, dx$$

$$th\frac{x}{2} = t \Rightarrow \frac{sh\frac{x}{2}}{ch\frac{x}{2}} = t$$

$$\frac{\frac{1}{2}ch^2\frac{x}{2} - \frac{1}{2}sh^2\frac{x}{2}}{ch^2\frac{x}{2}}\, dx = dt$$

$$\frac{1}{2ch^2\frac{x}{2}}\, dx = dt \Rightarrow \frac{1}{ch^2\frac{x}{2}} = 2dt$$

$$\frac{1}{ch^2\frac{x}{2}} = 1 - th^2\frac{x}{2}$$

$$th\frac{x}{2} = t \Rightarrow (1-t^2)\frac{dx}{2} = dt$$

$$dx = \frac{2dt}{1-t^2}$$

$$sh x = \frac{2t}{1-t^2} \qquad ch x = \frac{e^x - e^{-x}}{2}$$

التكاملات الأسية:

$$I = \int F(e^x)\, dx$$

$$e^x = t \Rightarrow e^x dx = dt \Rightarrow dx = \frac{dt}{t}$$

$$I = \int \frac{F(t)\, dt}{t}$$

مثال:

$$I = \int \frac{e^x - e^{-x} + 2}{e^x - e^{-x}}$$

SABBAGH STATIONERY