BASEL AL-HASSBANI

PhD Dissertation

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Stochastic Processes and Multivariate Permutation Statistics

Dissertation

zur Erlangung des Doktorgrades
der Fakultät für Mathematik, Informatik
und Naturwissenschaften
der Universität Hamburg

vorgelegt
im Department Mathematik
von

Basel Al-Hassbani

aus Daraa, Syrien

Hamburg

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Als Dissertation angenommen vom Fachbereich Mathematik der Universität Hamburg

auf Grund der Gutachten	**von**	**Prof. Dr. Georg Neuhaus**
	und	**Prof. Dr. Arnold Janssen**

Hamburg, den 25 November 2005

Prof. Dr. Alexander Kreuzer
Dekan des Fachbereichs Mathematik

Curriculum Vitae

Surname	Al-Hassbani
First Name	Basel
Gender	Male
Birth Date	06. January 1976
Birth Place	Daraa, Syria
Marital Status	Unmarried
Nationality	Syrian

Education and Research Programs

1981 - 1993	Elementary and High-School, Damascus
1993 - 1997	The University of Damascus
	Studying Pure Mathematics
	Obtaining Bachelor Degree in Pure Mathematics
1997 - 1998	Postgraduate Diploma Degree in Analysis
1998 - 2000	Assistant Teacher at The University of Damascus
2000 - 2002	The University of Hamburg
	Promotion Studies - Statistics
	Postgraduate Diploma Degree in Statistics
2002 - 2005	The University of Hamburg
	PhD Studies (Statistics) and Writing Dissertation
2006	Obtaining PhD Degree in Statistics

Language Skills	Arabic, English, German
Computer Literacy	Programming
Hobbies	Chess, Basketball Sport

TO MY LOVE

Contents:

Preface:

This doctoral thesis is intended mainly for considering the asymptotic normality of the distributions of certain permutation test statistics conditioned by some specific symmetric σ-fields. It is known that the classical theory of weak convergence has been the main base of the asymptotic distribution theorems, where these theorems have led to huge advance in statistics and probability in the last several decades. In particularly, this advance has led to obtain new powerful tests and to build the classical theory of statistical hypotheses testing, where the permutation tests have been developed also by depending on these theorems. And recently, after building the modern computers which have the capacity of making very accurate and fast computations, the permutation tests theory has become a big advantage in the modern theory of statistical tests. Moreover, the tremendous rise of the measure theory has led to enormous development in the theory of conditional expectations. And the results of the modern theory of conditional expectations are very promising in developing the classical theory of testing statistical hypotheses, where this thesis is a small step forward to build modern theory of statistical tests. And in fact, this thesis has been focused on some results in the theory of multivariate permutation tests, concerning the asymptotic normality of the conditional distributions of some linear permutation test statistics conditioned by some symmetric σ-fields, where also these σ-fields are generated by the considered random variables. Also, we introduced and generalized many theorems related with stochastic processes, and we put many central limit theorems and applications. This doctoral dissertation has three chapters. The first chapter is intended to give an exposition of many basic prerequisites about the conditional expectations, the conditional weak asymptotic equality, the conditional asymptotic normality, and the exchangeability. And this chapter has the definitions of these concepts in addition to many related theorems and other issues, and I mention here that some of these theorems and their proofs were built in parallel with similar ones in the book of Billingsley [1968], and also in the paper (On the asymptotic theory of permutation statistics) which is due to H. Strasser and C. Weber [1998]. In the second chapter we put several central limit theorems, related with stochastic processes. In the third chapter, I put some limit theorems and their proofs for the conditional distributions of linear test statistics under the familiar hypotheses H_0, H_1, H_2, and H_3. And this chapter contains four sections, where these sections were built to be similar to each other. And this is to give the reader a good vision about how to deal with similar hypotheses. The first section in this chapter has similar results to those of the mentioned paper above. Also, the second and the third chap-

ters give a vision of a new theory of testing statistical hypotheses, where the conditional expectations play the role of the integrals in the familiar classical theory of testing statistical hypotheses.
Finally, I wish to express my gratitude to Prof. Dr. Georg Neuhaus for his supervision and his help. I feel very much indebted to him for his efforts with me to make this work come true. I thank him deeply from my heart, I wish to him the best, and I will never forget him. I thank Prof. Dr. Arnold Janssen for reviewing my dissertation and his help. I thank Germany, and the staff of the University of Hamburg also the Studentenwerk in Hamburg, and the people of this land for their great and warm hospitality. I thank also my family very much, and all the friends who supported me. I thank the University of Damascus, and all staff of the Faculty of Sciences in Damascus, also the Syrian Government for funding my studies, and their great support to me.

Hamburg, June 2005.
BASEL AL-HASSBANI

Remarks:

Here we mention that definition 1.2.1, definition 1.3.1, definition 3.1.1, lemma 3.1.2, lemma 3.1.3, lemma 3.1.8, theorem 3.1.9, remark 3.1.10, definition 3.1.11, lemma 3.1.12, lemma 3.1.13, lemma 3.1.14, theorem 3.1.15, and remark 3.1.16 are mainly taken from the paper "On the asymptotic theory of permutation statistics" which is due to H. Strasser and C. Weber [1998].

Also, the ideas of definition 2.2.1, theorem 2.2.2, definition 2.2.3, theorem 2.2.4, theorem 2.2.5, theorem 2.2.6, and theorem 2.3.1 are essentially taken from the book "Convergence of Probability measures" which is due to Patrick Billingsley [1968].

List of symbols:

$\mathbb{N} = \{1, 2, 3, \cdots\}$	the set of all natural numbers.
$\mathbb{Q}$	the set of all rational numbers.
$\mathbb{R}$	the set of all real numbers.
$\mathbb{R}^d = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, d times.	
$\mathbb{B}^d$	the Borel σ-algebra defined on $\mathbb{R}^d$.
iff:	if and only if.
$E(f\|T)$, $E(f\|T = \cdot)$	see p. 9.
$E(X\|\mathcal{C})$	see p. 10.
$Var(X\|\mathcal{C})$	see p. 11.
$\vec{\bullet}$, $\overleftarrow{\bullet}$	see p. 12.
$\vec{\star}$	see p. 12.
`arr`	see p. 12.
$\parallel \cdot \parallel$	the usual norm defined on $\mathbb{R}^d$.
$\overset{w}{\sim}$ $(\mathcal{C}_n)$	see p. 16.
$F^{+\rho}$, $G^{-\rho}$, ∂G	see p. 17.
$\xrightarrow{P}$	tends in probability.
$\xrightarrow{\mathcal{D}}$	tends in distribution.
$[P]$	almost sure with respect to P.
$P_n * X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ $(\mathcal{C}_n)$	see p. 20.
$C[0, 1]$, $D[0, 1]$	see p. 32.
W, W^o	see p. 33.
$w(X_n, \delta)$	see p. 39.
ρ, σ^2	see p. 42.
$\exp(x) = e^x$	the exponential function.
$\triangle(h)$	see p. 46.
$c(v)$	see p. 47.
$\sigma(X_1, X_2, \cdots, X_k)$	the σ-field generated by $X_1, X_2, \cdots, X_k$.
$s^2(\underline{\xi}_{k_n})$, $\bar{\xi}_n$	see p. 86.
$\mathcal{S}^0(\underline{\xi}_{k_n})$	see p. 86.
$\mathcal{S}^0(\underline{\xi}_{k_n}, \mathcal{C})$	see p. 86.
π, Π_{k_n}	see p. 86.
$\parallel f \parallel_2 = (\int_0^1 \parallel f(t) \parallel^2 dt)^{\frac{1}{2}}$.	
e, Ξ_{k_n}	see p. 101.
$\underline{X, Y}_{k_n}$	see p. 101.
$\mathcal{S}^1(\underline{\xi, \xi}_{k_n})$	see p. 103.

Chapter One

Conditional Expectations

In this chapter we build the needed theoretical base of this research, where first we introduce concepts of the conditional expectations and the conditional variances in various cases and we define some binary operations to simplify the computations of the conditional expectations in case of random matrices. Also, we introduce a concept of conditional weak asymptotic equality and some related theorems which will play a major role in building the proofs of the invariance principles in chapter three. Also, we introduce a concept of conditional asymptotic normality and many needed results which form the essential arguments later. The main goal of this chapter is to make the reader familiar with the kinds of computations which are applied in the other two chapters. We mention here that many of the resultes and their proofs in this chapter are built on their similar ones in Billingsley [1968], and those in the paper (On the asymptotic theory of permutation statistics) which is due to H. Strasser and C. Weber [1998].

1.1. Conditional Expectation and Probability:

Introduction:

Let P be a probability measure defined over the sample space $(\Omega, \mathcal{A})$, T a statistic defined over $(\Omega, \mathcal{A})$ and taking values in some space $(\mathcal{J}, \mathcal{B})$, $\mathcal{A}_0$ the sub-σ-field it induces. Consider a nonnegative function f which is $\mathcal{A}$-measurable and P-integrable. Then $\int_A f dP$ is defined for $A \in \mathcal{A}$, and therefore for all $A_0 \in \mathcal{A}_0$. It follows from the Radon Nikodym theorem that there exists a function f_0 which is $\mathcal{A}_0$-measurable and P-integrable and such that

$$\int_{A_0} f dP = \int_{A_0} f_0 dP, \text{ for all } A_0 \in \mathcal{A}_0,$$

and that f_0 is unique with respect to $(\mathcal{A}_0, P)$, and it is known that there exists a $\mathcal{B}$-measurable function g such that

$$f_0 = g \circ T \ [P].$$

Let f_0 be denoted by $E(f|T)$ which is called the conditional expectation of f given T and g be denoted by $E(f|T = t)$ which is called the conditional expectation of f given $T = t$.

$$E(f|T) = E(f|T = \cdot) \circ T \ [P],$$

and easily we observe that:

$$\int_{T^{-1}(B)} f(x)dP(x) = \int_B E(f|T=t)dP^T(t).$$

So far, f has been assumed to be nonnegative. Without this condition, the conditional expectation of f given T is defined by

$$E(f|T) = E(f^+|T) - E(f^-|T)\ [P],$$

and the conditional expectation of f given $T=t$ is defined by

$$E(f|T=t) = E(f^+|T=t) - E(f^-|T=t)\ [P^T].$$

Also, that when the subtract operations are well-defined.

Definition 1.1.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space, and $\mathcal{A}_0$, a sub-σ-field of $\mathcal{A}$, and let $f : (\Omega, \mathcal{A}) \longrightarrow (\Sigma, \mathcal{B})$ be a measurable function. Then $E(f|\mathcal{A}_0) := E(f|id)$, where $id : (\Omega, \mathcal{A}) \longrightarrow (\Omega, \mathcal{A}_0)$, $id(x) = x$. $E(f|\mathcal{A}_0)$ is called the conditional expectation of f given $\mathcal{A}_0$.

Remark 1.1.2. The following properties of conditional expectation are needed in the coming arguments. Let $(\Omega, \mathcal{A}, P)$ be a probability space, X and Y be random variables on this space, i.e. $(X, Y \in \mathbb{R})$, $\mathcal{C} \subseteq \mathcal{A}$ be a sub-σ-field, and $\mathcal{A}_s \subseteq \mathcal{A}_t \subseteq \mathcal{A}$ be sub-σ-fields.

1. If $\mathcal{C} = \{\emptyset, \Omega\}$, then $E(X|\mathcal{C}) = EX\ [P]$.
2. $E(E(X|\mathcal{C})) = EX$.
3. $E(E(X|\mathcal{A}_s)|\mathcal{A}_t) = E(X|\mathcal{A}_s) = E(E(X|\mathcal{A}_t)|\mathcal{A}_s)\ [P]$.
4. If Y is $\mathcal{C}$-measurable then $E(XY|\mathcal{C}) = YE(X|\mathcal{C})\ [P]$.
5. $E(aX + bY|\mathcal{C}) = aE(X|\mathcal{C}) + bE(Y|\mathcal{C})\ [P]$, where a and b are constants.
6. If $g, f, f_1, f_2, \cdots, f_n, \cdots$ are $(\mathcal{A}, P)$-integrable, and $|f_n| \leqslant g$, $f_n \longrightarrow f\ [P]$, and $E(g|\mathcal{C})$ is well-defined then $E(f_n|\mathcal{C}) \longrightarrow E(f|\mathcal{C})\ [P]$.
7. $P(X \in B|\mathcal{C}) := E(1_{(X\in B)}|\mathcal{C})$ for all $B \in \mathbb{B}$.
8. X is independent of $Y \Longrightarrow E(X|Y) = EX\ [P]$.

9. $|E(X|\mathcal{C})| \leqslant E\left(|X|\middle|\mathcal{C}\right)$ $[P]$.

10. $E\left(\left|E\left(X|\mathcal{C}\right)\right|\right) \leqslant E\left(|X|\right)$.

11. $\mathcal{C} \subseteq \mathcal{C}_1 \Longrightarrow E\left(\left|E\left(X|\mathcal{C}\right)\right|\right) \leqslant E\left(\left|E\left(X|\mathcal{C}_1\right)\right|\right)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and $\mathcal{C}$ a sub-σ-field of $\mathcal{A}$, let further $X : \Omega \longrightarrow \mathbb{R}^d$ be a probability array, such that EX is well-defined. Then we define $E(X|\mathcal{C})$ by

$$E(X|\mathcal{C}) := \left(E(X^1|\mathcal{C}), \cdots, E(X^d|\mathcal{C})\right)^t,$$

where $X = (X^1, \cdots, X^d)^t$.
Also, if EX^iX^j, $i, j = 1, \cdots, d$, are well-defined, then

$$Var(X|\mathcal{C}) := E\left(\left(X - E(X|\mathcal{C})\right)\left(X - E(X|\mathcal{C})\right)^t|\mathcal{C}\right),$$

and we call it the conditional variance of X given $\mathcal{C}$.
Let $X : \Omega \longrightarrow \mathcal{M}_{d_1 \times d_2}$ be a random matrix such that EX^{ij}, $i = 1, \cdots, d_1$, $j = 1, \cdots, d_2$, are well-defined, then we define

$$E(X|\mathcal{C}) := \left(E(X^{ij}|\mathcal{C})\right)_{\substack{1 \leqslant i \leqslant d_1 \\ 1 \leqslant j \leqslant d_2}},$$

where $(X^{ij})_{\substack{1 \leqslant i \leqslant d_1 \\ 1 \leqslant j \leqslant d_2}}$.
And we define the random array $Y := \texttt{arr}(X) \in \mathbb{R}^{d_1 d_2}$ by

$$Y^1 := X^{11}, Y^2 := X^{21}, \cdots, Y^{d_1} := X^{d_1 1},$$

$$Y^{d_1+1} := X^{12}, Y^{d_1+2} := X^{22}, \cdots, Y^{2d_1} := X^{d_1 2},$$

$$\vdots$$

$$Y^{d_1(d_2-1)+1} := X^{1d_2}, \cdots, Y^{d_1 d_2} := X^{d_1 d_2}.$$

We mention here that, the function `arr` is also known by `vec`. This definition means that the column of Y is made from the columns of the matrix of X, which are taken from left to right in order.
Now we define $Var(X|\mathcal{C}) := Var(Y|\mathcal{C}) = Var(\texttt{arr}(X)|\mathcal{C})$.
Suppose that $A \in \mathcal{M}_{m \times d_1}$ is a given constant matrix, we want to give an

easy expression for $Var(AX|\mathcal{C})$.

$$Var(AX|\mathcal{C}) := Var(\texttt{arr}(AX)|\mathcal{C})$$
$$Var(AX|\mathcal{C}) = A \overset{\rightarrow}{\bullet} Var(X|\mathcal{C}) \overset{\leftarrow}{\bullet} A^t \ [P],$$

where $(X^{ij})_{\substack{1\leqslant i\leqslant d_1\\ 1\leqslant j\leqslant d_2}}$, $X^j := (X^{1j}, \cdots, X^{d_1 j})^t$, $j = 1, \cdots, d_2$, and

$$A \overset{\rightarrow}{\bullet} Var(X|\mathcal{C}) \overset{\leftarrow}{\bullet} A^t := \\ = \Big(A\ E\left(\left(X^{j_1} - E(X^{j_1}|\mathcal{C})\right)\left(X^{j_2} - E(X^{j_2}|\mathcal{C})\right)^t |\mathcal{C}\right) A^t\Big)_{1\leqslant j_1,j_2\leqslant d_2}.$$

From this we define the operations $\overset{\rightarrow}{\bullet}$, $\overset{\leftarrow}{\bullet}$ by the following:
Let $A \in \mathcal{M}_{m\times d_1}$, $B = (B^{ij})_{\substack{1\leqslant i\leqslant k_1\\ 1\leqslant j\leqslant k_2}}$, $B^{ij} \in \mathcal{M}_{d_1\times l}$, $i = 1, \cdots, k_1$, $j = 1, \cdots, k_2$, $C \in \mathcal{M}_{l\times m}$, then

$$A \overset{\rightarrow}{\bullet} B := (AB^{ij})_{\substack{1\leqslant i\leqslant k_1\\ 1\leqslant j\leqslant k_2}},$$

and

$$B \overset{\leftarrow}{\bullet} C := (B^{ij}C)_{\substack{1\leqslant i\leqslant k_1\\ 1\leqslant j\leqslant k_2}}.$$

Let $\xi : \Omega \longrightarrow \mathbb{R}^{d_1}$, and $\eta : \Omega \longrightarrow \mathbb{R}^{d_2}$, be random arrays such that

$$E(\xi^{i_1}\xi^{i_2}\eta^{j_1}\eta^{j_2}),\ 1 \leqslant i_1, i_2 \leqslant d_1,\ 1 \leqslant j_1, j_2 \leqslant d_2,$$

are well-defined and satisfying that

$$E(\xi^{i_1}\xi^{i_2}\eta^{j_1}\eta^{j_2}|\mathcal{C}) = E(\xi^{i_1}\xi^{i_2}|\mathcal{C})E(\eta^{j_1}\eta^{j_2}|\mathcal{C}) \ [P],$$

for all $1 \leqslant i_1, i_2 \leqslant d_1$, and all $1 \leqslant j_1, j_2 \leqslant d_2$, then we easily see that

$$E\left(\texttt{arr}(\xi\eta^t) \cdot (\texttt{arr}(\xi\eta^t))^t|\mathcal{C}\right) = E(\xi\xi^t|\mathcal{C}) \overset{\rightarrow}{\star} E(\eta\eta^t|\mathcal{C}) \ [P],$$

where

$$E(\xi\xi^t|\mathcal{C}) \overset{\rightarrow}{\star} E(\eta\eta^t|\mathcal{C}) := \left(E(\xi\xi^t|\mathcal{C})E(\eta^{j_1}\eta^{j_2}|\mathcal{C})\right)_{1\leqslant j_1,j_2\leqslant d_2}.$$

From this, we define the operation $\overset{\rightarrow}{\star}$ by the following:
Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{l\times k}$, then we define $A \overset{\rightarrow}{\star} B := (A \cdot b^{ij})_{\substack{1\leqslant i\leqslant l\\ 1\leqslant j\leqslant k}}$, where $B = (b^{ij})_{\substack{1\leqslant i\leqslant l\\ 1\leqslant j\leqslant k}}$.

Theorem 1.1.3. Let $(\Omega, \mathcal{A}, P)$ be a probability space, and $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence of sub-σ-fields of $\mathcal{A}$, let further $\rho_n : \Omega \longrightarrow \mathbb{R}$ be non-negative random variables, such that $E(\rho_n|\mathcal{C}_n) \xrightarrow{P} 0$. Then $\rho_n \xrightarrow{P} 0$ also.

Proof: From the hypotheses we find that the limit $P\{|E(\rho_n|\mathcal{C}_n)| \geqslant \gamma\} \underset{n\longrightarrow\infty}{\longrightarrow} 0$ is valid for all $\gamma > 0$.
We note from the definition of conditional expectation also that

$$\int_{\{|E(\rho_n|\mathcal{C}_n)|<\gamma\}} \rho_n dP \leqslant \gamma, \forall\gamma > 0.$$

Let $\varepsilon, \delta > 0$ be arbitrary and fixed temporarily. For $\gamma = \varepsilon\delta$, we have

$$P\left\{\left|\rho_n 1_{\{|E(\rho_n|\mathcal{C}_n)|<\gamma\}}\right| \geqslant \delta\right\} \leqslant \frac{1}{\delta} E\left(\rho_n 1_{\{|E(\rho_n|\mathcal{C}_n)|<\gamma\}}\right) \leqslant \varepsilon,$$

$$P\left\{\left|\rho_n 1_{\{|E(\rho_n|\mathcal{C}_n)|\geqslant\gamma\}}\right| \geqslant \delta\right\} \leqslant P\left\{|E(\rho_n|\mathcal{C}_n)| \geqslant \gamma\right\}.$$

Also, we conclude that $P\{|\rho_n| \geqslant \delta\} \underset{n\longrightarrow\infty}{\longrightarrow} 0,\ \forall\delta > 0.$ □

The next definition gives a concept of the stochastical boundedness, which will play a big role in the rest of this research.

Definition 1.1.4. Let X_n, $n \in \mathbb{N}$, be random elements defined on $(\Omega, \mathcal{A}, P)$ and taking values in the norm space $(S, \|\cdot\|)$. We call the sequence $(X_n)_{n\in\mathbb{N}}$ stochastically bounded iff $\forall\varepsilon > 0 : \exists M > 0$, $P\{\| X_n \|\leqslant M\} \geqslant 1-\varepsilon$, is valid for all $n \in \mathbb{N}$. □

In the same situation of theorem 1.1.3, but here we assume that $(E(\rho_n|\mathcal{C}_n))_{n\in\mathbb{N}}$ is stochastically bounded. Then we have that the sequence $(\rho_n)_{n\in\mathbb{N}}$ is also stochastically bounded.
Let $\varepsilon > 0$, there exists $M > 0$ such that

$$P\left\{\left|\rho_n 1_{\{|E(\rho_n|\mathcal{C}_n)|>M\}}\right| > \frac{2M}{\varepsilon}\right\} \leqslant P\left\{|E(\rho_n|\mathcal{C}_n)| > M\right\} \leqslant \frac{\varepsilon}{2},$$

$$P\left\{\left|\rho_n 1_{\{|E(\rho_n|\mathcal{C}_n)|\leqslant M\}}\right| > \frac{2M}{\varepsilon}\right\} \leqslant \frac{\varepsilon}{2M} E\left(\rho_n 1_{\{|E(\rho_n|\mathcal{C}_n)|\leqslant M\}}\right) \leqslant \frac{\varepsilon}{2},$$

$$P\left\{|\rho_n| > \frac{2M}{\varepsilon}\right\} \leqslant \varepsilon,$$

$$P\left\{|\rho_n| \leqslant M'\right\} \geqslant 1-\varepsilon \text{ , where } M' = \frac{2M}{\varepsilon}.$$

□

Let us here introduce here a general concept of exchangeablity, also we shall present some related arguments, which will be needed later in chapter three.
Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\underline{\xi}_k = (\xi_1, \xi_2, \cdots, \xi_k)^t$ be a triangular array of random elements from $(\Omega, \mathcal{A}, P)$ to $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. Let further G be a finite group of measurable functions $h : \mathcal{E}^k \longrightarrow \mathcal{E}^k$, and $\mathcal{C}$ be a sub-σ-field of $\mathcal{A}$. Finally, let $\mathcal{S}(\xi_k) := \left(\underline{\xi}_k\right)^{-1}(\mathcal{S}_k)$, where $\mathcal{S}_k$ denotes here the σ-field of all sets B in $\mathcal{B}^k$ satisfying the condition

$$\underline{x}_k \in B \Longleftrightarrow h\left(\underline{x}_k\right) \in B, \ \forall h \in G,$$

where here $\underline{x}_k := (x_1, \cdots, x_k)$.

Definition 1.1.5. The random arrays $\xi_1,\ \xi_2,\ \cdots,\ \xi_k$ are G-exchangeable under $P(\cdot|\mathcal{C})$, iff $\forall h \in G$ the equality

$$P\left(\underline{\xi}_k \in B|\mathcal{C}\right) = P\left(h\left(\underline{\xi}_k\right) \in B|\mathcal{C}\right) \ [P]$$

is valid for all $B \in \mathcal{B}^k$, and then the array $\underline{\xi}_k$ is called G-exchangeable under $P(\cdot|\mathcal{C})$.

Lemma 1.1.6. Let $f : (\mathcal{E}, \mathcal{B})^k \longrightarrow (\mathbb{R}, \mathbb{B})$ be a measurable function such that $E(f(\underline{\xi}_k))$ is well-defined. If $\underline{\xi}_k$ is G-exchangeable under $P(\cdot|\mathcal{C})$, then $\forall A \in \mathcal{S}(\underline{\xi}_k),\ C \in \mathcal{C},\ h \in G$ the following equality is valid

$$E\left(1_{A\cap C} f\left(h\left(\underline{\xi}_k\right)\right)\right) = E\left(1_{A\cap C} f\left(\underline{\xi}_k\right)\right).$$

Proof: Let $h \in G,\ A \in \mathcal{S}(\underline{\xi}_k)$, and $C \in \mathcal{C}$. Then there is $\tilde{A} \in \mathcal{B}^k$ satisfying $A = \left(\underline{\xi}_k\right)^{-1}(\tilde{A})$ and $h(\tilde{A}) = \tilde{A}$. Consequently, we have $1_{\tilde{A}}(h(\underline{\xi}_k)) = 1_{\tilde{A}}(\underline{\xi}_k)$. We shall prove the assertion only for $f = 1_{\tilde{B}}$, where $\tilde{B} \in \mathcal{B}^k$, since the general case is a direct result of using the monotone convergence theorem for conditional expectations.
$E\big[1_{A\cap C} 1_{\tilde{B}}(h(\underline{\xi}_k))\big]$
$= E\big[1_{\tilde{A}}(\underline{\xi}_k) 1_{\tilde{B}}(h(\underline{\xi}_k)) 1_C\big]$
$= E\big[1_{\tilde{A}}(h(\underline{\xi}_k)) 1_{\tilde{B}}(h(\underline{\xi}_k)) 1_C\big]$
$= E\big[1_{\tilde{A}\cap\tilde{B}}(h(\underline{\xi}_k)) 1_C\big]$

$$
\begin{aligned}
&= E\big[E\big\{1_{\tilde{A}\cap\tilde{B}}(h(\underline{\xi}_k))|\mathcal{C}\big\}1_C\big] \\
&= E\big[E\big\{1_{\tilde{A}\cap\tilde{B}}(\underline{\xi}_k)|\mathcal{C}\big\}1_C\big] \\
&= E\big[1_{\tilde{A}\cap\tilde{B}}(\underline{\xi}_k)1_C\big] \\
&= E\big[1_{A\cap C}1_{\tilde{B}}(\underline{\xi}_k)\big]. \qquad \square
\end{aligned}
$$

Let us denote the σ-field $\sigma(\mathcal{S}(\underline{\xi}_k), \mathcal{C})$ by $\mathcal{S}(\underline{\xi}_k, \mathcal{C})$. Now, since $\{A \cap C : A \in \mathcal{S}(\underline{\xi}_k),\ C \in \mathcal{C}\}$ generates the σ-field $\mathcal{S}(\underline{\xi}_k, \mathcal{C})$, we have by lemma 1.1.6 the validity of

$$
E\left(1_D f\left(h\left(\underline{\xi}_k\right)\right)\right) = E\left(1_D f\left(\underline{\xi}_k\right)\right),
$$

$\forall D \in \mathcal{S}(\underline{\xi}_k, \mathcal{C}),\ h \in G$.
Therefore, the following lemma is just a consequence of lemma 1.1.6.

Lemma 1.1.7. Let $f : (\mathcal{E}, \mathcal{B})^k \longrightarrow (\mathbb{R}, \mathbb{B})$ be a measurable function such that $E(f(\underline{\xi}_k))$ is well-defined. If $\underline{\xi}_k$ is G-exchangeable under $P(\cdot|\mathcal{C})$, then

$$
E(f(\underline{\xi}_k)|\mathcal{S}(\underline{\xi}_k, \mathcal{C})) = \frac{1}{|G|}\sum_{h\in G} f(h(\underline{\xi}_k))\ [P],
$$

where $|G|$ is the size of the finite group G.

1.2. Conditional Weak Asymptotic Equality:

In this section, we introduce a concept of weakly asymptotically equal distributions conditioned by given sub-σ-fields. The following definition determines this concept.

Definition 1.2.1. Assume that S is a metric space, $\mathcal{S}$ is the Borel σ-field, $\mathbb{F}$ is the set of all bounded uniformly continuous functions $f : S \longrightarrow \mathbb{R}$. Let further $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ be sequences of random elements from $(\Omega, \mathcal{A}, P)$ to S, and let $(\mathcal{C}_n)_{n\in\mathbb{N}}$, $\mathcal{C}_n \subseteq \mathcal{A}$, be a sequence of sub-$\sigma$-fields. We say that the sequences $(X_n)_{n\in\mathbb{N}}$, and $(Y_n)_{n\in\mathbb{N}}$ are weakly asymptotically equal conditioned by $(\mathcal{C}_n)_{n\in\mathbb{N}}$, and write that by symbols $X_n \overset{w}{\sim} Y_n(\mathcal{C}_n)$, iff $E(f(X_n)|\mathcal{C}_n) - E(f(Y_n)|\mathcal{C}_n) \xrightarrow{P} 0$, is valid for all $f \in \mathbb{F}$.
The following lemma provides a useful criterion for weak asymptotic equality.

Lemma 1.2.2. The sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are weakly asymptotic equal iff the following limit

$$\int_{C_n} f(X_n)dP - \int_{C_n} f(Y_n)dP \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

is valid for all $(C_n)_{n\in\mathbb{N}}$, where $C_n \in \mathcal{C}_n$, $n \in \mathbb{N}$, and for all $f \in \mathbb{F}$.

Proof: Since f is bounded, the validity of the limit follows from the dominated convergence theorem. Now, let us prove the sufficiency. Suppose temporarily the contrary, then there is $f \in \mathbb{F}$, a sub-sequence $\{n'\}$ of $\{n\}$, $\varepsilon > 0$, and $\delta > 0$, such that for $C_{n'} \in \mathcal{C}_{n'}$, defined by

$$C_{n'} := \{\omega : \omega \in S,\ |(E(f(X_{n'})|\mathcal{C}_{n'}) - E(f(Y_{n'})|\mathcal{C}_{n'}))(\omega)| \geqslant \varepsilon\},\ \forall n \in \mathbb{N},$$

we have $P(C_{n'}) \geqslant \delta$, $n = 1, 2, \cdots$, and consequently there exists a sub-subsequence $\{n''\}$ of $\{n'\}$ such that either

$$C_{n''} = \{\omega : \omega \in S,\ (E(f(X_{n''})|\mathcal{C}_{n''}) - E(f(Y_{n''})|\mathcal{C}_{n''}))(\omega) \geqslant \varepsilon\},$$

or

$$C_{n''} = \{\omega : \omega \in S,\ (E(f(Y_{n''})|\mathcal{C}_{n''}) - E(f(X_{n''})|\mathcal{C}_{n''}))(\omega) \geqslant \varepsilon\},$$

is valid for all $n'' \in \mathbb{N}$.
But in the first case we find

$$\int_{C_{n''}} f(X_{n''})dP - \int_{C_{n''}} f(Y_{n''})dP = \int_{C_{n''}} E\left(f(X_{n''}) - f(Y_{n''})|\mathcal{C}_{n''}\right)dP,$$

$\Longrightarrow$

$$\int_{C_{n''}} f(X_{n''})dP - \int_{C_{n''}} f(Y_{n''})dP \geqslant \varepsilon\delta > 0, \ \forall n'' \in \mathbb{N},$$

and in the second case similarly also

$$\int_{C_{n''}} f(Y_{n''})dP - \int_{C_{n''}} f(X_{n''})dP \geqslant \varepsilon\delta > 0, \ \forall n'' \in \mathbb{N}.$$

And this is a contradiction clearly. Hence, the sufficiency holds. □

The next theorem provides equivalent conditions to weak asymptotic equality, any one of these conditions could serve as a definition.

Therem 1.2.3. Assume the same situation of definition 1.2.1, then the following conditions are equivalent.

(i) $X_n \overset{w}{\sim} Y_n \ (\mathcal{C}_n)$.

(ii) $E(f(X_n)|\mathcal{C}_n) - E(f(Y_n)|\mathcal{C}_n) \overset{P}{\longrightarrow} 0$, is valid for all bounded Lipschitz-continuous functions $f : S \longrightarrow \mathbb{R}$.

(iii) $P\left(E\left(1_F(X_n)|\mathcal{C}_n\right) - E\left(1_{F^{+\rho}}(Y_n)|\mathcal{C}_n\right) \geqslant \varepsilon\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0, \ \forall \rho > 0, \ \forall \varepsilon > 0$, where F is an arbitrary closed set in S, $F^{+\rho} := \{x : x \in S, \ d(x, F) \leqslant \rho\}$, and 1_F, $1_{F^{+\rho}}$ are indicator functions.

(iv) $P\left(E\left(1_G(X_n)|\mathcal{C}_n\right) - E\left(1_{G^{-\rho}}(Y_n)|\mathcal{C}_n\right) \leqslant -\varepsilon\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0, \ \forall \rho > 0, \ \forall \varepsilon > 0$, where G is an arbitrary open set in S, $G^{-\rho} := \{x : x \in G, \ d(x, \partial G) > \rho\}$, and 1_G, $1_{G^{-\rho}}$ are indicator functions.

Proof: We shall prove this theorem as the following diagram

(i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv)$\Longrightarrow$(iii)$\Longrightarrow$ (i).

Of course, (i)$\Longrightarrow$(ii) is trivial because of the simple fact that any Lipschitz-continuous function is uniformly continuous.

Proof of (ii)$\Longrightarrow$(iii).

Suppose that (ii) holds and that F is closed. Suppose further that $\rho > 0$. Let $f : S \longrightarrow \mathbb{R}$ be defined by $f(x) := \varphi\left(\frac{1}{\rho}d(x, F)\right)$, where $\varphi(t)$ is defined by $\varphi(t) := \begin{cases} 1 & : \quad t \leqslant 0 \\ 1 - t & : \quad 0 \leqslant t \leqslant 1 \\ 0 & : \quad 1 \leqslant t \end{cases}$. Then f is a Lipschitz-continuous function,

and also (ii) holds for this f. It is easy to see that $1_F \leqslant f \leqslant 1_{F^{+\rho}}$. Hence, for all $n \in \mathbb{N}$, we have $E\left(1_F(X_n)|\mathcal{C}_n\right) - E\left(1_{F^{+\rho}}(Y_n)|\mathcal{C}_n\right) \leqslant E\left(f(X_n)|\mathcal{C}_n\right) - E\left(f(Y_n)|\mathcal{C}_n\right)$ $[P]$. Also, the assertion follows.
The equivalence (iii) $\Longleftrightarrow$ (iv) follows by complementation.
Proof of (iii)$\Longrightarrow$(i).
Suppose that (iii) holds and that $f : S \longrightarrow \mathbb{R}$ is bounded uniformly continuous. we shall show that $E(f(X_n)|\mathcal{C}_n) - E(f(Y_n)|\mathcal{C}_n) \xrightarrow{P} 0$. First, we shall prove that $P(E(f(X_n)|\mathcal{C}_n) - E(f(Y_n)|\mathcal{C}_n) \geqslant \varepsilon) \underset{n\longrightarrow\infty}{\longrightarrow} 0$, $\forall\varepsilon > 0$. Without any loss of generality, we can assume that $0 \leqslant f(x) \leqslant q < 1$, $\forall x \in S$, where q is fixed. In general, we may reduce the problem to this case by transforming f linearly. For an integral k, temporarily fixed, let F_i be the closed set $F_i := \left\{x : x \in S,\ \frac{i}{k} \leqslant f(x)\right\}$, $i = 1, 2, \cdots, k$. Since $0 \leqslant f(x) \leqslant q < 1$, $\forall x \in S$, we have for all $\varepsilon > 0$ which satisfying $q < 1 - \varepsilon$

$$\sum_{i=1}^{k} \left(\tfrac{i-1}{k} - \varepsilon\right) \cdot 1_{\{t:\frac{i-1}{k}-\varepsilon\leqslant f(t)<\frac{i}{k}-\varepsilon\}} \leqslant \sum_{i=1}^{k} \tfrac{i}{k} \cdot 1_{\{t:\frac{i-1}{k}\leqslant f(t)<\frac{i}{k}\}}.$$

This can be rewritten as the following

$$\left(\tfrac{1}{k}\sum_{i=1}^{k} 1_{\{t\in S:\frac{i}{k}-\varepsilon\leqslant f(t)\}}\right) - \varepsilon 1_S \leqslant f < \tfrac{1}{k}1_S + \tfrac{1}{k}\sum_{i=1}^{k} 1_{F_i}.$$

Since f is uniformly continuous, for $\varepsilon > 0$ there is $\rho > 0$ such that $F_i^{+\rho} \subseteq \{x : x \in S,\ \frac{i}{k} - \varepsilon \leqslant f(x)\}$, $i = 1, 2, \cdots, k$.

Hence, $\left(\frac{1}{k}\sum_{i=1}^{k} 1_{F_i^{+\rho}}\right) - \varepsilon 1_S \leqslant f < \frac{1}{k}1_S + \frac{1}{k}\sum_{i=1}^{k} 1_{F_i}$.

Also, $E(f(X_n)|\mathcal{C}_n) < \frac{1}{k}1_S + \frac{1}{k}\sum_{i=1}^{k} E(1_{F_i}(X_n)|\mathcal{C}_n)$ $[P]$, and

$-E(f(Y_n)|\mathcal{C}_n) \leqslant \varepsilon 1_S - \frac{1}{k}\sum_{i=1}^{k} E(1_{F_i^{+\rho}}(Y_n)|\mathcal{C}_n)$ $[P]$.

Hence,

$$E(f(X_n)|\mathcal{C}_n) - E(f(Y_n)|\mathcal{C}_n) \leqslant$$
$$\leqslant \left(\tfrac{1}{k} + \varepsilon\right) 1_S + \tfrac{1}{k}\sum_{i=1}^{k} \Big(E(1_{F_i}(X_n)|\mathcal{C}_n) - E(1_{F_i^{+\rho}}(Y_n)|\mathcal{C}_n)\Big) \ [P].$$

Thus, $P\left(E(f(X_n)|\mathcal{C}_n) - E(f(Y_n)|\mathcal{C}_n) \geqslant \varepsilon\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0$, $\forall\varepsilon > 0$, indeed. Putting $-f$ in place of f, we obtain $P\left(E(f(X_n)|\mathcal{C}_n) - E(f(Y_n)|\mathcal{C}_n) \leqslant -\varepsilon\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0$.

Therefore, $E(f(X_n)|\mathcal{C}_n) - E(f(Y_n)|\mathcal{C}_n) \xrightarrow{P} 0$. □

For the next lemma suppose that X_n, Y_n, Z_n, $n \in \mathbb{N}$, are random elements from $(\Omega, \mathcal{A}, P)$ to the separable metric space (S, d).

Lemma 1.2.4. If $X_n \overset{w}{\sim} Z_n$ $(\mathcal{C}_n)$ and $d(X_n, Y_n) \xrightarrow{P} 0$, then $Y_n \overset{w}{\sim} Z_n$

$(\mathcal{C}_n)$.

Proof: We shall show that the following implication $E(f(X_n)|\mathcal{C}_n) - E(f(Z_n)|\mathcal{C}_n) \xrightarrow{P} 0,\ d(X_n, Y_n) \xrightarrow{P} 0 \Longrightarrow E(f(Y_n)|\mathcal{C}_n) - E(f(Z_n)|\mathcal{C}_n) \xrightarrow{P} 0$, is valid for all $f \in \mathbb{F}$. It is clear that $f(X_n) - f(Y_n) \xrightarrow{P} 0$, and $E(f(X_n)|\mathcal{C}_n) - E(f(Y_n)|\mathcal{C}_n) \xrightarrow{P} 0$, also $E(f(Y_n)|\mathcal{C}_n) - E(f(Z_n)|\mathcal{C}_n) \xrightarrow{P} 0$.
Hence, the assertion holds. □

1.3. Conditional Asymptotic Normality:

This section contains a concept of conditional asymptotic normality of sequence of random arrays. This concept is created to discuss a certain behavior of the conditional distributions of a given sequence of random elements.

Definition 1.3.1. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random arrays defined on a measurable space $(\Omega, \mathcal{A})$ and taking values in $\mathbb{R}^k$, and let $(P_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on that space. Let further $(\mathcal{C}_n)_{n\in\mathbb{N}}$, $\mathcal{C}_n \subseteq \mathcal{A}$, be a sequence of sub-$\sigma$-fields, and consider the sequences $(\mu_n)_{n\in\mathbb{N}}$, $(\sigma_n^2)_{n\in\mathbb{N}}$ where for each $n \in \mathbb{N}$, μ_n and σ_n^2 are a random array and a random matrix respectively, and they are defined on $(\Omega, \mathcal{C}_n)$ and taking values in $\mathbb{R}^k$ and $\mathcal{M}_k$ respectively. We say that the sequence $(X_n)_{n\in\mathbb{N}}$ is asymptotically normal under $(P_n)_{n\in\mathbb{N}}$ and conditioned by $(\mathcal{C}_n)_{n\in\mathbb{N}}$ if the limit

$$E_{P_n}(f(X_n)|\mathcal{C}_n) - \int f d\mathcal{N}(\mu_n, \sigma_n^2) \xrightarrow{P_n} 0,$$

is valid for all bounded uniformly continuous functions $f : \mathbb{R}^k \longrightarrow \mathbb{R}$, and we denote that by

$$P_n * X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)\ (\mathcal{C}_n).$$

The next theorem provides equivalent conditions to the concept of the asymptotic normality, any one of these conditions could serve as a definition.

Theorem 1.3.2. Assume the same situation of definition 1.3.1 then the following conditions are equivalent.

(i) $P_n * X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)\ (\mathcal{C}_n)$.

(ii) $E_{P_n}(f(X_n)|\mathcal{C}_n) - \int f d\mathcal{N}(\mu_n, \sigma_n^2) \xrightarrow{P_n} 0$, is valid for all bounded Lipschitz continuous functions $f : \mathbb{R}^k \longrightarrow \mathbb{R}$.

(iii) $P_n \left(E_{P_n}(1_F(X_n)|\mathcal{C}_n) - \int 1_{F^{+\rho}} d\mathcal{N}(\mu_n, \sigma_n^2) \geqslant \varepsilon \right) \underset{n\longrightarrow\infty}{\longrightarrow} 0$, $\forall \rho > 0$, $\forall \varepsilon > 0$, where F is an arbitrary closed set in $\mathbb{R}^k$, and 1_F, $1_{F^{+\rho}}$ are defined as in theorem 1.2.3.

(iv) $P_n \left(E_{P_n}(1_G(X_n)|\mathcal{C}_n) - \int 1_{G^{-\rho}} d\mathcal{N}(\mu_n, \sigma_n^2) \leqslant -\varepsilon \right) \underset{n\longrightarrow\infty}{\longrightarrow} 0$, $\forall \rho > 0$, $\forall \varepsilon > 0$, where G is an arbitrary open set in $\mathbb{R}^k$, and 1_G, $1_{G^{-\rho}}$ are defined as in theorem 1.2.3.

Proof: It is easy to see that this theorem is similar to theorem 1.2.3, also the proof here can be built similarly. □

For the next lemma suppose that X_n, Y_n, Z_n, $n \in \mathbb{N}$, are random arrays from $(\Omega, \mathcal{A}, P)$ to $(\mathbb{R}^k, \|\cdot\|)$, where $\|\cdot\|$ is the usual norm on $\mathbb{R}^k$, and $\mathbb{F}$ is the set of all bounded uniformly continuous functions $f : \mathbb{R}^k \longrightarrow \mathbb{R}$.

Lemma 1.3.3. If $P_n * X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ $(\mathcal{C}_n)$, and if $\| X_n - Y_n \| \xrightarrow{P_n} 0$, then we have $P_n * Y_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ $(\mathcal{C}_n)$.

Proof: Since $\| X_n - Y_n \| \xrightarrow{P_n} 0$, we obtain that $f(X_n) - f(Y_n) \xrightarrow{P_n} 0$ is valid for all $f \in \mathbb{F}$. This implies $E(f(X_n)|\mathcal{C}_n) - E(f(Y_n)|\mathcal{C}_n) \xrightarrow{P_n} 0$ is valid for all $f \in \mathbb{F}$, and also $E_{P_n}(f(Y_n)|\mathcal{C}_n) - \int f d\mathcal{N}(\mu_n, \sigma_n^2) \xrightarrow{P_n} 0$ is valid for all $f \in \mathbb{F}$. □

Theorem 1.3.4. Let X_n, $n \in \mathbb{N}$, be a random array from $(\Omega, \mathcal{A}, P)$ to $\mathbb{R}^m$, assume that $P * X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ $(\mathcal{C}_n)$, and assume further that for each $n \in \mathbb{N}$, Y_n is a $\mathcal{C}_n$-measurable and stochastically bounded m-dimensional array, then

$$P * (X_n + Y_n) \sim \mathcal{N}(\mu_n + Y_n, \sigma_n^2) \ (\mathcal{C}_n).$$

Proof: It is sufficient to prove it for the case when there exists $M > 0$ such that $P(\| Y_n \| < M) = 1 \ \forall n \in \mathbb{N}$.
Since one can reduce the general case to this one by using $Y_n 1_{\{\|Y_n\|<M\}}$ instead of Y_n. And then we get the validity of

$$P * \left(X_n + Y_n 1_{\{\|Y_n\|<M\}}\right) \sim \mathcal{N}\left(\mu_n + Y_n 1_{\{\|Y_n\|<M\}}, \sigma_n^2\right) \ (\mathcal{C}_n).$$

Which implies

$$P\left\{\left|E\left(f\left(X_n + Y_n\right)|\mathcal{C}_n\right) - \int f d\mathcal{N}(\mu_n + Y_n, \sigma_n^2)\right| > \gamma, \| Y_n \| \leqslant M\right\} \underset{n\to\infty}{\longrightarrow} 0,$$

for all $\gamma > 0$. and we note from the hypotheses $P\{\| Y_n \| > M\} < \varepsilon$, and this will complete the proof. Now, we prove the assertion in the case $P(\| Y_n \| < M) = 1 \ \forall n \in \mathbb{N}$. Let $\mathbb{F}$ be the set of all bounded uniformly continuous functions $f : \mathbb{R}^m \longrightarrow \mathbb{R}$. We want to prove the validity of the following implication

$E\left(f(X_n)|\mathcal{C}_n\right) - \int f d\mathcal{N}(\mu_n,\sigma_n^2) \xrightarrow{P} 0,\ \forall f \in \mathbb{F} \Longrightarrow E\left(f(X_n+Y_n)|\mathcal{C}_n\right) - \int f d\mathcal{N}(\mu_n+Y_n,\sigma_n^2) \xrightarrow{P} 0,\ \forall f \in \mathbb{F}$.
Let $a \in \mathbb{R}^m$ be a given constant and let g be defined as $g(\cdot) := f(\cdot + a)$. Then $g \in \mathbb{F}$, and we have
$E\left(g(X_n)|\mathcal{C}_n\right) - \int g d\mathcal{N}(\mu_n,\sigma_n^2) \xrightarrow{P} 0$, which implies
$E\left(f(X_n+a)|\mathcal{C}_n\right) - \int f(\cdot+a) d\mathcal{N}(\mu_n,\sigma_n^2) \xrightarrow{P} 0,\ \forall a \in \mathbb{R}^m$. But this means
$P\left(\left|E\left(f(X_n+a)|\mathcal{C}_n\right) - \int f(\cdot+a) d\mathcal{N}(\mu_n,\sigma_n^2)\right| \geqslant \varepsilon\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0,\ \forall \varepsilon > 0,\ \forall a \in \mathbb{R}^m$. Let us define the $\mathcal{C}_n$-measurable functions $h_{k,n} := \left(h_{k,n}^1,\cdots,h_{k,n}^m\right)^t : \Omega \longrightarrow \mathbb{R}^m$ by $h_{k,n}^j := \sum_{i=0}^{k-1}\left(-M+\frac{2M}{k}i\right)1_{\{-M+\frac{2M}{k}i\leqslant\pi_j(Y_n)<-M+\frac{2M}{k}(i+1)\}}$, $j = 1,\cdots,m$, where π_1, π_2, $\cdots$, π_m are the usual projection functions. It is easy to see that $\parallel Y_n - h_{k,n} \parallel \leqslant \frac{2M}{k}\sqrt{m}$, and

$$h_{k,n}(\omega) \in \left\{ \left(-M+\frac{2M}{k}i_1,\cdots,-M+\frac{2M}{k}i_m\right)^t : \right.$$
$$\left. : (i_1,\cdots,i_m) \in \{0,\cdots,k-1\}^m \right\},$$

$\forall \omega \in \Omega$. Let us prove the validity of the limit
$P\left(\left|E\left(f(X_n+h_{k,n})|\mathcal{C}_n\right) - \int f(\cdot+h_{k,n}) d\mathcal{N}(\mu_n,\sigma_n^2)\right| \geqslant \varepsilon\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0,\ \forall \varepsilon > 0,\ \forall k \in \mathbb{N}$.
For this let us define the constants

$$c_{i_1,\cdots,i_m} := \left(-M+\frac{2M}{k}i_1,\cdots,-M+\frac{2M}{k}i_m\right)^t.$$

Now, $P\left(\left|E\left(f(X_n+h_{k,n})|\mathcal{C}_n\right) - \int f(\cdot+h_{k,n}) d\mathcal{N}(\mu_n,\sigma_n^2)\right| \geqslant \varepsilon\right) =$

$$\sum_{(i_1,\cdots,i_m)\in\{0,\cdots,k-1\}^m} P\left(h_{k,n} = c_{i_1,\cdots,i_m},\ \left|E\left(f(X_n+c_{i_1,\cdots,i_m})|\mathcal{C}_n\right) - \right.\right.$$
$$\left.\left. - \int f(\cdot+c_{i_1,\cdots,i_m}) d\mathcal{N}(\mu_n,\sigma_n^2)\right| \geqslant \varepsilon\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

$\forall \varepsilon > 0,\ \forall k \in \mathbb{N}$.
Since f is uniformly continuous, for a given $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $P\left(\left|E\left(f(X_n+Y_n)|\mathcal{C}_n\right) - \int f(\cdot+Y_n) d\mathcal{N}(\mu_n,\sigma_n^2)\right| \geqslant \varepsilon\right) \leqslant$

$$\leqslant P\left(\left|E\left(f(X_n+h_{k,n})|\mathcal{C}_n\right) - \int f(\cdot+h_{k,n}) d\mathcal{N}(\mu_n,\sigma_n^2)\right| \geqslant \varepsilon/2\right),\ \forall n \in \mathbb{N}.$$

Thus,
$P\left(\left|E\left(f(X_n+Y_n)|\mathcal{C}_n\right) - \int f(\cdot+Y_n) d\mathcal{N}(\mu_n,\sigma_n^2)\right| \geqslant \varepsilon\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0$. But this is

equivalent to
$P\left(\left|E\left(f(X_n+Y_n)|\mathcal{C}_n\right)-\int f d\mathcal{N}(\mu_n+Y_n,\sigma_n^2)\right|\geqslant\varepsilon\right)\underset{n\longrightarrow\infty}{\longrightarrow}0$. Since $\varepsilon>0$ is arbitrary, the assertion follows. □

Theorem 1.3.5. Assume the same situation of definition 1.3.1, but here $P_n=P$, $\forall n\in\mathbb{N}$. Assume further that there exist $M>0$, $\gamma>0$ such that for each $n\in\mathbb{N}$ we have $P\{\parallel\mu_n\parallel\leqslant M\}=1$, $P\{\parallel\sigma_n^2\parallel\leqslant M\}=1$, $P\{|\det(\sigma_n^2)|\geqslant\gamma\}=1$. Then the following conditions are equivalent:

(i) $E(f(X_n)|\mathcal{C}_n)-\int f d\mathcal{N}(\mu_n,\sigma_n^2)\overset{P}{\longrightarrow}0$, for all bounded continuous functions $f:\mathbb{R}^k\longrightarrow\mathbb{R}$.

(ii) $P*X_n\sim\mathcal{N}(\mu_n,\sigma_n^2)\ (\mathcal{C}_n)$.

(iii) Let F be a closed set in $\mathbb{R}^k$, G be an open set in $\mathbb{R}^k$, and let $\{n^{(1)}\}$ be any subsequence of $\{n\}$, then there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N\subset\Omega$, such that $P(N)=0$, and for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$, and for each $\omega\in\Omega-N$ there exists a sub-subsequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that the following inequalities are well-defined and valid

$$\limsup_{n^{(4)}} E(1_F(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega)\leqslant\lim_{n^{(4)}}\int 1_F d\mathcal{N}(\mu_{n^{(4)}}(\omega),\sigma^2_{n^{(4)}}(\omega)),$$

and

$$\liminf_{n^{(4)}} E(1_G(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega)\geqslant\lim_{n^{(4)}}\int 1_G d\mathcal{N}(\mu_{n^{(4)}}(\omega),\sigma^2_{n^{(4)}}(\omega)).$$

(iv) $E(1_A(X_n)|\mathcal{C}_n)-\int 1_A d\mathcal{N}(\mu_n,\sigma_n^2)\overset{P}{\longrightarrow}0$, for all $\mathcal{N}(0,I_k)$-continuity sets $A\subseteq\mathbb{R}^k$.

Proof: Before we start with the steps of this proof, we mention here that in the condtion (iii) the sequence $\{n^{(4)}\}$ depends on the element $\omega\in\Omega$ also the set N depends on the sequence $\{n^{(1)}\}$ also F and/or G.
Now, similarly to theorem 1.2.3, and by computations which are familiar from the portmanteau theorem in Billingsley [1968].
The implication $(i)\Longrightarrow(ii)$ is clear.
Proof of $(ii)\Longrightarrow(iii)$.

Suppose that (ii) is valid and that F is a closed set in $\mathbb{R}^k$, and let $\{n^{(1)}\}$ be any subsequence of $\{n\}$. Then there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N \subset \Omega$, $P(N) = 0$, such that the limit
$E(f_\rho(X_{n^{(2)}})|\mathcal{C}_{n^{(2)}})(\omega) - \int f_\rho d\mathcal{N}(\mu_{n^{(2)}}(\omega), \sigma^2_{n^{(2)}}(\omega)) \underset{n^{(2)}\longrightarrow\infty}{\longrightarrow} 0,$
is valid $\forall\omega \in \Omega - N$, $\forall\rho \in \mathbb{N}$, where $f_\rho : \mathbb{R}^k \longrightarrow \mathbb{R}$, $f_\rho(x) := \varphi(\rho d(x, F))$, and where $\varphi(t) := \begin{cases} 1 & : \quad t \leqslant 0 \\ 1-t & : \quad 0 \leqslant t \leqslant 1 \\ 0 & : \quad 1 \leqslant t \end{cases}.$
And also $\parallel \mu_{n^{(2)}}(\omega) \parallel \leqslant M$, $\parallel \sigma^2_{n^{(2)}}(\omega) \parallel \leqslant M$, and $|\det(\sigma^2_{n^{(2)}}(\omega))| \geqslant \gamma$ are valid for all $n \in \mathbb{N}$, $\omega \in \Omega - N$. Consequently, for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$ and for any fixed $\omega \in \Omega - N$ there exists a sub-sequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that the sequences $\{\mu_{n^{(4)}}(\omega)\}$, $\{\sigma^2_{n^{(4)}}(\omega)\}$ are convergent. And we have in the same time the validity of the limit
$E(f_\rho(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) - \int f_\rho d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \underset{n^{(4)}\longrightarrow\infty}{\longrightarrow} 0$, $\forall\rho \in \mathbb{N}$. And we see that
$E(1_F(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) \leqslant E(f_\rho(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega)$
$\limsup\limits_{n^{(4)}} E(1_F(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) \leqslant \lim\limits_{n^{(4)}} \int f_\rho d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega))$, $\forall\rho \in \mathbb{N}$.
Thus,
$\limsup\limits_{n^{(4)}} E(1_F(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) \leqslant \lim\limits_{n^{(4)}} \int 1_F d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega))$.
And by complementary, we conclude also for $G = F^c$ the following inequality.
$\liminf\limits_{n^{(4)}} E(1_G(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) \geqslant \lim\limits_{n^{(4)}} \int 1_G d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega))$.
Moreover, we can easily conclude that the condition (iii) in this theorem is equivalent to the following one
Let $\{F_1, F_2, \cdots\}$ be a countable family of closed sets in $\mathbb{R}^k$, $\{G_1, G_2, \cdots\}$ be a countable family of open sets in $\mathbb{R}^k$, and let $\{n^{(1)}\}$ be any subsequence of $\{n\}$, then there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N \subset \Omega$, such that $P(N) = 0$, and for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$, and for each $\omega \in \Omega - N$ there exists a sub-subsequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that

$$\limsup_{n^{(4)}} E(1_F(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) \leqslant \lim_{n^{(4)}} \int 1_F d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)),$$

and

$$\liminf_{n^{(4)}} E(1_G(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) \geqslant \lim_{n^{(4)}} \int 1_G d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)),$$

for all $F \in \{F_1, F_2, \cdots\}$ and also for all $G \in \{G_1, G_2, \cdots\}$.
Let us now prove the implication $(iii) \Longrightarrow (iv)$.

Let A^o denote the interior of $A \subseteq \mathbb{R}^k$, and let A^- denote its closure. If (iii) is valid, and let $\{n^{(1)}\}$ be any subsequence of $\{n\}$, then there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N \subset \Omega$, such that $P(N) = 0$, and for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$, and for each $\omega \in \Omega - N$ there exists a sub-subsequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that the sequences $\{\mu_{n^{(4)}}(\omega)\}$, $\{\sigma^2_{n^{(4)}}(\omega)\}$ are convergent, and we have
$\lim_{n^{(4)}} \int 1_{A^-} d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \geqslant \limsup_{n^{(4)}} E(1_{A^-}(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega)$
$\geqslant \limsup_{n^{(4)}} E(1_A(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega)$
$\geqslant \liminf_{n^{(4)}} E(1_A(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega)$
$\geqslant \liminf_{n^{(4)}} E(1_{A^o}(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega)$
$\geqslant \lim_{n^{(4)}} \int 1_{A^o} d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega))$.
If A is $\mathcal{N}(0, I_k)$- continuity set, then $\lim_{n^{(4)}} \int 1_{\partial A} d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) = 0$.
Consequently,

$$E(1_A(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) - \int 1_A d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \underset{n^{(4)} \longrightarrow \infty}{\longrightarrow} 0.$$

And therefore, $\forall \omega \in \Omega - N$ we have

$$E(1_A(X_{n^{(2)}})|\mathcal{C}_{n^{(2)}})(\omega) - \int 1_A d\mathcal{N}(\mu_{n^{(2)}}(\omega), \sigma^2_{n^{(2)}}(\omega)) \underset{n^{(2)} \longrightarrow \infty}{\longrightarrow} 0,$$

which implies that

$$E(1_A(X_n)|\mathcal{C}_n) - \int 1_A d\mathcal{N}(\mu_n, \sigma^2_n) \xrightarrow{P} 0.$$

Moreover, we can easily see that the condition (iv) in this theorem is equivalent to the following one.
Let $\{A_1, A_2, \cdots\}$ be a countable family of $\mathcal{N}(0, I_k)$-continuity sets in $\mathbb{R}^k$, and let $\{n^{(1)}\}$ be any subsequence of $\{n\}$, then there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N \subset \Omega$, such that $P(N) = 0$, and for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$, and for each $\omega \in \Omega - N$ there exists a sub-subsequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that

$$E(1_A(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) - \int 1_A d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \underset{n^{(4)} \longrightarrow \infty}{\longrightarrow} 0,$$

$\forall A \in \{A_1, A_2, \cdots\}$.
Proof of $(iv) \Longrightarrow (iii)$.
Since $\partial\{x : d(x, F) \leqslant \delta\}$ is contained in $\{x : d(x, F) = \delta\}$, we find that these

boundaries are disjoint for distinct δ's, and then there exists a sequence $\{\delta_j\}$, such that $\delta_j \underset{j\longrightarrow\infty}{\longrightarrow} 0$, and the sets $F_j = \{x, d(x,F) \leqslant \delta_j\}$, $j \in \mathbb{N}$. If (iv) is valid then for $\{F_1, F_2, \cdots\}$. And let $\{n^{(1)}\}$ be any subsequence of $\{n\}$, then there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N \subset \Omega$, such that $P(N) = 0$, and for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$, and for each $\omega \in \Omega - N$ there exists a sub-subsequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that
$E(1_{F_j}(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) - \int 1_{F_j} d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \underset{n^{(4)}\longrightarrow\infty}{\longrightarrow} 0$, $\forall j \in \mathbb{N}$.
Consequently, by routine and easy computations, as in page 14 in Billingsley [1968], we have the validity of (iii) for F, and by complementation, we have also the validity of (iii) for G.
Proof of $(iii) \Longrightarrow (i)$.
It is sufficient to prove this implication in the case where $0 \leqslant f < 1$, and f is continuous. Let us define $F_{ji} := \{x : i/j \leqslant f(x)\}$, $i = 1, 2, \cdots, j$.
And consequently,
$\{F_{ji},\ i = 1, 2, \cdots;\ j = 1, 2, \cdots\}$ is a countable family of closed sets in $\mathbb{R}^k$.
Let $\mathbb{P}$ be a probability measure defined on $\mathbb{R}^k$, then we have
$\sum_{i=1}^{j} \frac{i-1}{j}\mathbb{P}\{x : \frac{i-1}{j} \leqslant f(x) < \frac{i}{j}\} \leqslant \int f d\mathbb{P} < \sum_{i=1}^{j} \frac{i}{j}\mathbb{P}\{x : \frac{i-1}{j} \leqslant f(x) < \frac{i}{j}\}$.
And consequently, we have
$\frac{1}{j}\sum_{i=1}^{j} \mathbb{P}(F_{ji}) \leqslant \int f d\mathbb{P} < \frac{1}{j} + \frac{1}{j}\sum_{i=1}^{j} \mathbb{P}(F_{ji})$,
If (iii) is valid, and let $\{n^{(1)}\}$ be any subsequence of $\{n\}$, then there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N \subset \Omega$, such that $P(N) = 0$, and for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$, and for each $\omega \in \Omega - N$ there exists a sub-subsequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that the sequences $\{\mu_{n^{(4)}}(\omega)\}$, $\{\sigma^2_{n^{(4)}}(\omega)\}$ are convergent and

$$\limsup_{n^{(4)}} E(1_F(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) \leqslant \lim_{n^{(4)}} \int 1_F d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)),$$

for all $F \in \{F_{ji},\ i = 1, 2, \cdots;\ j = 1, 2, \cdots\}$.
Moreover, we have
$\limsup_{n^{(4)}} E(f(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) < \frac{1}{j} + \frac{1}{j}\sum_{i=1}^{j} \lim_{n^{(4)}} \int 1_{F_{ji}} d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \leqslant$
$\leqslant \frac{1}{j} + \lim_{n^{(4)}} \int f d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega))$.
By letting $j \longrightarrow \infty$, we obtain
$\limsup_{n^{(4)}} E(f(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) \leqslant \lim_{n^{(4)}} \int f d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega))$,
we replace now f by $-f$, and then we obtain
$\liminf_{n^{(4)}} E(f(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) \geqslant \lim_{n^{(4)}} \int f d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega))$.

Therefore, we have
$E(f(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}})(\omega) - \int f d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \underset{n^{(4)}\longrightarrow\infty}{\longrightarrow} 0.$
The proof is complete. □

Corollary 1.3.6. Similarly as we did at the beginning of the proof of theorem 1.3.4, here also in theorem 1.3.5 we can release the conditions imposed on the sequences $(\mu_n)_{n\in\mathbb{N}}$, $(\sigma_n^2)_{n\in\mathbb{N}}$ where we assume here that these sequences are stochastically bounded, and for each $n \in \mathbb{N}$, μ_n, σ_n^2 are $\mathcal{C}_n$-measurable. We assume further that $\forall\varepsilon > 0 : \exists\gamma > 0$, $\exists n_0 \in \mathbb{N}$, such that

$$P\{|\det(\sigma_n^2)| \geqslant \gamma\} \geqslant 1 - \varepsilon, \ \forall n \geqslant n_0.$$

Then the conditions (i), (ii), (iv) in theorem 1.3.5 are equivalent.

Theorem 1.3.7. Assume the same situation of corollary 1.3.6. If $P * X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ $(\mathcal{C}_n)$, and let $h : \mathbb{R}^k \longrightarrow \mathbb{R}$ be measurable and the set of discontinuities of h is negligible with respect to $\mathcal{N}(0, I_k)$, then $P * h(X_n) \sim \mathcal{N}(\mu_n, \sigma_n^2)h^{-1}$ $(\mathcal{C}_n)$ or more explicitly

$$E(f(h(X_n))|\mathcal{C}_n) - \int f(h(\cdot))d\mathcal{N}(\mu_n, \sigma_n^2) \xrightarrow{P} 0,$$

for all bounded uniformly continuous functions $f : \mathbb{R}^k \longrightarrow \mathbb{R}$.

Proof: In a similar way as theorem 5.1 in Billingsley [1968], theorem 1.3.5, and corollary 1.3.6, it is sufficient here to prove the assertion in the case where for suitable $M > 0$, $\gamma > 0$, and $\forall n \in \mathbb{N}$, we have $P\{\| \mu_n \| \leqslant M\} = 1$, $P\{\| \sigma_n^2 \| \leqslant M\} = 1$, $P\{|\det(\sigma_n^2)| \geqslant \gamma\} = 1$.
We want to prove that for a given closed set F in $\mathbb{R}^k$, and a subsequence $\{n^{(1)}\}$ of $\{n\}$, there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N \subset \Omega$, such that $P(N) = 0$, and for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$, and for each $\omega \in \Omega - N$ there exists a sub-subsequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that the following inequality is well-defined and valid

$$\limsup_{n^{(4)}} E\left(1_F(h(X_{n^{(4)}}))|\mathcal{C}_{n^{(4)}}\right)(\omega) \leqslant \lim_{n^{(4)}} \int 1_F(h(\cdot))d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)).$$

For this, let $(h^{-1}(F))^{-}$ be the closure set of $h^{-1}(F)$. And let us denote the set of discontinuities of h by D_h. From the hypotheses D_h is negligible with respect to $\mathcal{N}(0, I_k)$ and $(h^{-1}(F))^{-} \subseteq D_h \cup h^{-1}(F)$.

Let $\{n^{(1)}\}$ be a subsequence of $\{n\}$, then there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N \subset \Omega$, such that $P(N) = 0$, and for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$, and for each $\omega \in \Omega - N$ there exists a sub-subsequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that the following computations are valid

$$\begin{aligned}
&\limsup_{n^{(4)}} E\left(1_F(h(X_{n^{(4)}}))|\mathcal{C}_{n^{(4)}}\right)(\omega) = \\
&= \limsup_{n^{(4)}} E\left(1_{h^{-1}(F)}(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}}\right)(\omega) \\
&\leqslant \limsup_{n^{(4)}} E\left(1_{(h^{-1}(F))^-}(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}}\right)(\omega) \\
&\leqslant \lim_{n^{(4)}} \int 1_{(h^{-1}(F))^-} d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \\
&= \lim_{n^{(4)}} \int 1_{h^{-1}(F)} d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \\
&= \lim_{n^{(4)}} \int 1_F(h(\cdot)) d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)).
\end{aligned}$$

□

Let h_n, $n \in \mathbb{N}$, and h be measurable functions from $\mathbb{R}^k$ to $\mathbb{R}$, and let D be the set of all $x \in \mathbb{R}^k$, such that there exists a sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in \mathbb{R}^k$, $\forall n \in \mathbb{N}$ and $x_n \longrightarrow x$, but $h_n(x_n) \longrightarrow h(x)$ fails to hold.
Then it is known that this D is $\mathcal{N}(0, I_k)$-measurable.

Theorem 1.3.8. In the same situation of corollary 1.3.6, and if $P * X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ $(\mathcal{C}_n)$, and $\mathcal{N}(0, I_k)(D) = 0$, then $P * h_n(X_n) \sim \mathcal{N}(\mu_n, \sigma_n^2)h_n^{-1}$ $(\mathcal{C}_n)$, or more explicitly

$$E(f(h_n(X_n))|\mathcal{C}_n) - \int f(h_n(\cdot))d\mathcal{N}(\mu_n, \sigma_n^2) \xrightarrow{P} 0,$$

for all bounded uniformly continuous functions $f : \mathbb{R}^k \longrightarrow \mathbb{R}$.

Proof: It is sufficient here to prove the assertion in the case where for suitable $M > 0$, $\gamma > 0$, and $\forall n \in \mathbb{N}$, we have $P\{\| \mu_n \| \leqslant M\} = 1$, $P\{\| \sigma_n^2 \| \leqslant M\} = 1$, $P\{|\det(\sigma_n^2)| \geqslant \gamma\} = 1$. By applying theorem 1.3.5, and computations which are similar from theorem 5.5 in Billingsley [1968] we conclude the validity of this theorem.
And for this, we want to prove that for a given open set G in $\mathbb{R}^k$, and a subsequence $\{n^{(1)}\}$ of $\{n\}$, there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N \subset \Omega$, such that $P(N) = 0$, and for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$, and for each $\omega \in \Omega - N$ there exists a sub-subsequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that the following inequality is well-defined and valid

$$\liminf_{n^{(4)}} E\left(1_G(h_{n^{(4)}}(X_{n^{(4)}}))|\mathcal{C}_{n^{(4)}}\right)(\omega) \geqslant \lim_{n^{(4)}} \int 1_G(h_{n^{(4)}}(\cdot))d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)).$$

Since D is negligible with respect to $\mathcal{N}(0, I_k)$,
we find that G is a $\lim_{n^{(4)}} \mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega))h^{-1}_{n^{(4)}}$-continuity set, and it is sufficient to prove

$$\liminf_{n^{(4)}} E\left(1_G(h_{n^{(4)}}(X_{n^{(4)}}))|\mathcal{C}_{n^{(4)}}\right)(\omega) \geqslant \lim_{n^{(4)}} \int 1_G(h(\cdot))d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)).$$

For this, let $T_l := \cap_{i \geqslant l} h_i^{-1}(G)$, and for each $l \in \mathbb{N}$ let T_l^o be the interior set of T_l.
We note that $h^{-1}(G) \subseteq D \cup_{l=1}^{\infty} T_l^o$. Let $\{n^{(1)}\}$ be a subsequence of $\{n\}$, then there exists a sub-subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $N \subset \Omega$, such that $P(N) = 0$, and for any subsequence $\{n^{(3)}\}$ of $\{n^{(2)}\}$, and for each $\omega \in \Omega - N$ there exists a sub-subsequence $\{n^{(4)}\}$ of $\{n^{(3)}\}$ such that the following computations are valid
$\lim_{n^{(4)}} \int 1_D d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) = 0$
$\lim_{n^{(4)}} \int 1_G(h(\cdot))d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) =$
$= \lim_{n^{(4)}} \int 1_{h^{-1}(G)} d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega))$
$= \lim_{n^{(4)}} \int 1_{\cup_{l=1}^{\infty} T_l^o} d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega))$.
Let $\varepsilon > 0$. Since $T_l^o \subseteq T_{l+1}^o$, for $l = 1, 2, \cdots$, we have
$\lim_{n^{(4)}} \int 1_G(h(\cdot))d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \leqslant$
$\leqslant \lim_{n^{(4)}} \int 1_{T_l^o} d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) + \varepsilon$, for sufficiently large l.
$\leqslant \liminf_{n^{(4)}} E\left(1_{T_l^o}(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}}\right)(\omega) + \varepsilon$
$\leqslant \liminf_{n^{(4)}} E\left(1_{h^{-1}_{n^{(4)}}(G)}(X_{n^{(4)}})|\mathcal{C}_{n^{(4)}}\right)(\omega) + \varepsilon$
$\leqslant \liminf_{n^{(4)}} E\left(1_G\left(h_{n^{(4)}}(X_{n^{(4)}})\right)|\mathcal{C}_{n^{(4)}}\right)(\omega) + \varepsilon$.
We conclude finally the validity of the inequality
$\lim_{n^{(4)}} \int 1_G(h(\cdot))d\mathcal{N}(\mu_{n^{(4)}}(\omega), \sigma^2_{n^{(4)}}(\omega)) \leqslant \liminf_{n^{(4)}} E\left(1_G\left(h_{n^{(4)}}(X_{n^{(4)}})\right)|\mathcal{C}_{n^{(4)}}\right)(\omega)$.
The proof is complete $\square$

Chapter Two

Stochastic Processes

In this chapter we shall discuss some stochastic processes which satisfy certain characteristics. Where we focus here on the d-dimensional Brownian motions and also the d-dimensional Brownian bridges. We introduce some central limit theorems and we put some applications. All these issues will be used later in chapter three in building the results related with multivariate permutation test statistics. And we mention here that, the conditional distribution limit theorems which are introduced in chapter three can be considered as direct consequences of the applications of this chapter. In the same way, one can derive many limit theorems in the field of permutation test statistics. And actually, one can build a comprehensive theory where the conditional expectations play the role of the ordinary expectations in the classical theory of testing hypotheses. And we have just opened a way for that in this dissertation.

2.1. Gaussian Processes and Some Essential Concepts:

We begin with the well-known concept of a general process. Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(\mathcal{S}, \mathcal{B})$ an arbitrary measurable space, and $\mathcal{T}$ an arbitrary set. The family of measurable functions $(X_\tau)_{\tau\in\mathcal{T}}$ is called a stochastic process, defined on $(\Omega, \mathcal{A}, P)$, with state space $(\mathcal{S}, \mathcal{B})$, and index set $\mathcal{T}$.
Also, one can see that, if $\mathcal{T}$ is the set all positive integer numbers $\{ 1, 2, 3, \cdots \}$, and $\mathcal{S} = \mathbb{R}$, the set of all real numbers, and $\mathcal{B} = \mathbb{B}$ the Borel σ-field on $\mathbb{R}$, then $(X_\tau)_{\tau\in\mathcal{T}}$ is the sequence of random variables $X_1, X_2, \cdots$. And if $\mathcal{T} = \{ 1, 2, \cdots, d \}$, then we obtain a d-dimensional random array. Also a general process can be considered as a single random mapping X from Ω to $\mathcal{S}^{\mathcal{T}}$, the set of all functions from $\mathcal{T}$ to $\mathcal{S}$, see R. Ash [1975].
$X(\omega) := \{X_\tau(\omega) : \tau \in \mathcal{T}\}$, and $\mathcal{S}^{\mathcal{T}}$ is provided with the σ-field $\mathcal{A}^{\mathcal{T}}$, where

$$\mathcal{A}^{\mathcal{T}} := \sigma\Bigg(\big\{\varpi \in \mathcal{S}^{\mathcal{T}} : \varpi(\tau_1) \in B_1, \cdots, \varpi(\tau_m) \in B_m\big\} : \\ : \ \tau_1, \cdots, \tau_m \in \mathcal{T}; \text{ and } B_1, \cdots, B_m \in \mathcal{B}, \text{ for } m \in \mathbb{N}\Bigg).$$

Now, we shall introduce the concept of the d-dimensional Gaussian processes.
If $\mathcal{S} = \mathbb{R}^d$, $\mathcal{B} = \mathbb{B}^d$, and $\mathcal{T} = \mathbb{R}$, then we call X a d-dimensional Gaussian

process iff for each $t_1,\ t_2,\ \cdots,\ t_n \in \mathcal{T}$, and for $n = 1,\ 2,\ \cdots$, the random array $(X(t_1), \cdots, X(t_n))^t$ has normal distribution.
Before we introduce the d-dimensional Brownian processes, we begin with the definitions of the spaces $(C[0,1])^d$, $(D[0,1])^d$, and we focus on some of their features.
$(C[0,1])^d$ is the space of all continuous functions from $[0,1]$ to $\mathbb{R}^d$, provided with the metric

$$\mathcal{D}(x,y) := \sup_{0\leqslant t\leqslant 1} \|x(t)-y(t)\|,\ \forall x,y \in (C[0,1])^d$$

where

$$\|\gamma\| := \max_{1\leqslant l\leqslant d}|\gamma^l|,\ \forall \gamma := (\gamma^1, \cdots, \gamma^d)^t \in \mathbb{R}^d.$$

The topology generated by this metric is called the uniform topology, and it is known that, the space $(C[0,1])^d$ is complete and separable under this metric.

$(D[0,1])^d$ is the space of all functions x defined on $[0,1]$ and taking values in $\mathbb{R}^d$, such that x is right-continuous and has left-hand limit at any point $t \in [0,1]$.

$$\lim_{t\downarrow t_0} x(t) = x(t_0),\ \forall t_0 \in [0,1),$$

and

$$\lim_{t\uparrow t_0} x(t),\text{ is exist, } \forall t_0 \in (0,1].$$

And it is known that, there can be at most finitely many points t at which $||x(t) - \lim\limits_{t\uparrow t_0} x(t)|| > \varepsilon$, for a given $\varepsilon > 0$. Moreover, x has at most countably many discontinuities, and it is bounded.

$$\sup_{0\leqslant t\leqslant 1} ||x(t)|| < \infty.$$

Let Λ be the set of all strictly increasing, continuous functions $\lambda : [0,1] \longrightarrow [0,1]$. If $\lambda \in \Lambda$, then $\lambda(0) = 0$ and $\lambda(1) = 1$.
We define the metric here $\mathcal{D}$ as the following

$$\mathcal{D}(x,y) = \inf\left\{\varepsilon : \sup_{0\leqslant t\leqslant 1}|\lambda(t)-t| \leqslant \varepsilon,\ \sup_{0\leqslant t\leqslant 1}||x(t)-y(\lambda(t))|| \leqslant \varepsilon\right\},$$

$$,\ \forall x,y \in (C[0,1])^d.$$

It is known that the topology generated by this metric is called the Skorohod topology. And the space $(D[0,1])^d$ is separable but not complete under this $\mathcal{D}$. It exists another metric $\mathcal{D}_0$ which generates the same topology but the space $(D[0,1])^d$ is separable and complete under $\mathcal{D}_0$.

$$\mathcal{D}_0(x,y) = \inf\left\{\varepsilon : \sup_{0\leqslant s<t\leqslant 1}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| \leqslant \varepsilon,\ \sup_{0\leqslant t\leqslant 1}\|x(t)-y(\lambda(t))\| \leqslant \varepsilon\right\}, \quad \forall x,y \in (D[0,1])^d.$$

For more details see Billingsley [1968].

We are now at the suitable position to introduce the concept of the d-dimensional Brownian processes.

We define a d-dimensional Brownian motion W as an array $\left(W^1,\ W^2,\ \cdots,\ W^d\right)^t$ where $W^1,\ W^2,\ \cdots,\ W^d$ are independent one-dimensional Brownian motions. Let μ be the Wiener measure, defined on the space $(C[0,1])^d$, then the identity function W defined as the following

$$W : (C[0,1])^d \longrightarrow (C[0,1])^d$$
$$W(\omega) = \omega,\ \forall \omega \in (C[0,1])^d$$

is called the standard d-dimensional Brownian motion. And we shall use the symbols W, and μ when we want to deal with d-dimensional Brownian motion and Wiener measure, defined on $(C[0,1])^d$.

Also similarly to case when $d = 1$, here we define the standard d-dimensional Brownian bridge W^o as the following

$$W^o : (C[0,1])^d \longrightarrow (C[0,1])^d$$

$$W^o(\omega) = \tilde{\omega},\ \forall \omega \in (C[0,1])^d,$$

and where

$$\tilde{\omega}(t) := \omega(t) - t\omega(1).$$

And consequently,

$$(W^o_t)_{0\leqslant t\leqslant 1} \sim (W_t - tW_1)_{0\leqslant t\leqslant 1}.$$

Where $W_t(\omega) := \omega(t),\ W^o_t(\omega) := \tilde{\omega}(t),\ \forall \omega \in (C[0,1])^d$, and $\forall t \in [0,1]$.

And in words we say that, the processes $(W^o_t)_{0\leqslant t\leqslant 1}$ and $(W_t - tW_1)_{0\leqslant t\leqslant 1}$ are equivalent.

2.2. Central Limit Theorems:

In this section, we generalize some of the theorems in chapter four in Billingsley [1968]. Where we prove these theorems in the case of d dimensions. One of these theorems characterizing d-dimensional Brownian motions, and another one characterizing more general d-dimensional Gaussian processes. Also, we introduce asymptotic forms of these two theorems. We mention here that one can use these theorems to build many applications as we shall see in the next section.

Definition 2.2.1. Let X be a random element defined on $(\Omega, \mathcal{A}, P)$ and taking values in $(D[0,1])^d$. We say that X has independent increments iff for all partitions of $[0,1]$

$$0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant 1, \ k \in \mathbb{N}.$$

Then the increments

$$X(t_1) - X(t_0), \ X(t_2) - X(t_1), \cdots, X(t_k) - X(t_{k-1})$$

are independent under P.

Theorem 2.2.2. Let X be a random element defined on $(\Omega, \mathcal{A}, P)$ and taking values in $(D[0,1])^d$ such that it has independent increments and satisfies

$$P\{X \in (C[0,1])^d\} = 1, \ E(X(t)) = 0,$$

and

$$E\left(X(t)(X(t))^t\right) = tI_d, \ \forall t \in [0,1].$$

Then X is distributed as W, where W here is a d-dimensional Brownian motion. In symbols $X \sim W$.

Proof: Since X has independent increments, we have for $s \leqslant t$, and $l_1, l_2 \in \{1, 2, \cdots, d\}$, $X^{l_2}(s)$, and $\left(X^{l_1}(t) - X^{l_1}(s)\right)$ are independent under P, and from the assumptions imposed on X, we find also

$$E\left(X^{l_1}(t)X^{l_2}(s)\right) = E\left(X^{l_2}(s)\left(X^{l_1}(t) - X^{l_1}(s)\right)\right) + E\left(X^{l_1}(s)X^{l_2}(s)\right)$$

$$= 0 + E\left(X^{l_1}(s)X^{l_2}(s)\right) = \begin{cases} 0 & : \ l_1 \neq l_2 \\ s & : \ l_1 = l_2 \end{cases}$$

Consequently, for general s, t, taken from $[0,1]$, we have

$$E\left(X^{l_1}(t)X^{l_2}(s)\right) = \begin{cases} 0 & : \ l_1 \neq l_2 \\ \min(t,s) & : \ l_1 = l_2 \end{cases}.$$

And this will lead to

$$E\left(X(t)\left(X(s)\right)^t\right) = \min(t,s)I_d,\ \forall t,s \in [0,1],$$

$$E\left(X(\tilde{t})\left(X(\tilde{s})\right)^t\right) = \min{}^*(\tilde{t},\tilde{s}),\ \forall \tilde{t},\tilde{s} \in ([0,1])^d,$$

where

$$\tilde{t} := (t_1, t_2, \cdots, t_d)^t,$$
$$\tilde{s} := (s_1, s_2, \cdots, s_d)^t,$$
$$X(\tilde{t}) := \left(X^1(t_1), X^2(t_2), \cdots, X^d(t_d)\right)^t,$$
$$X(\tilde{s}) := \left(X^1(s_1), X^2(s_2), \cdots, X^d(s_d)\right)^t,$$
$$\min{}^*(\tilde{t},\tilde{s}) := \begin{pmatrix} \min(t_1,s_1) & 0 & \cdots & 0 \\ 0 & \min(t_2,s_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \min(t_d,s_d) \end{pmatrix},$$
$$E\left(\left(X(t)-X(s)\right)\left(\left(X(t)-X(s)\right)^t\right) = |t-s|I_d,\ \forall t,s \in [0,1],$$

and from the assumptions we conclude also that $P\{X(0)=0\}=1$.
Now, we shall show that to prove the validity of the assertion of this theorem, it is sufficient to prove that $X(t)-X(s)$ has normal distribution $\mathcal{N}(0,(t-s)I_d)$ for $s \leqslant t$. Since X has independent increments, this will imply that the random array $(X(t_1), X(t_2), \cdots, X(t_k))^t$ for $t_1,\ t_2,\ \cdots,\ t_k \in [0,1]$, has normal distribution $\mathcal{N}(0,\Gamma)$, where

$$\Gamma := \begin{pmatrix} t_1 I_d & \min(t_1,t_2)I_d & \cdots & \min(t_1,t_k)I_d \\ \min(t_2,t_1)I_d & t_2 I_d & \cdots & \min(t_2,t_k)I_d \\ \vdots & \vdots & \ddots & \vdots \\ \min(t_k,t_1)I_d & \min(t_k,t_2)I_d & \cdots & t_k I_d \end{pmatrix}.$$

So far, we found that if we prove that $X(t)-X(s)$, for $s \leqslant t$, has normal distribution, then this will lead to the fact that all finite-dimensional distributions of X and W are identical, and consequently $X \sim W$ indeed.
And actually, if $X(t)$, for $t \in [0,1]$, has normal distribution $\mathcal{N}(0, tI_d)$, then $X(t)-X(s)$, for $s \leqslant t$, has normal distribution $\mathcal{N}(0,(t-s)I_d)$. And this comes from the fact that $X(s)$, $X(t)-X(s)$ are independent and from the following equalities

$$E\left(e^{i\tilde{u}^t X(t)}\right) = E\left(e^{i\tilde{u}^t (X(t)-X(s))}\right) E\left(e^{i\tilde{u}^t X(s)}\right),\ \forall \tilde{u} \in \mathbb{R}^d,$$

$$E\left(e^{i\tilde{u}^t X(t)}\right) = \exp(-\frac{1}{2}\tilde{u}^t t I_d \tilde{u}),\ \forall \tilde{u} \in \mathbb{R}^d,$$
$$E\left(e^{i\tilde{u}^t X(s)}\right) = \exp(-\frac{1}{2}\tilde{u}^t s I_d \tilde{u}),\ \forall \tilde{u} \in \mathbb{R}^d,$$

and consequently

$$E\left(e^{i\tilde{u}^t (X(t)-X(s))}\right) = \exp(-\frac{1}{2}\tilde{u}^t (t-s) I_d \tilde{u}),\ \forall \tilde{u} \in \mathbb{R}^d.$$

Now, since $X \in (C[0,1])^d$ with probability one, we find $e^{i\tilde{u}^t X(t)} \underset{t\longrightarrow 1}{\longrightarrow} e^{i\tilde{u}^t X(1)}$ almost sure for all $\omega \in \Omega$. And since $\left|e^{i\tilde{u}^t X(t)}\right| = 1,\ \forall t \in [0,1]$, we conclude that

$$E\left(e^{i\tilde{u}^t X(t)}\right) \underset{t\longrightarrow 1}{\longrightarrow} E\left(e^{i\tilde{u}^t X(1)}\right).$$

Therefore, to achieve our goal here it is truly sufficient to prove that $X(t)$, for $t < 1$, has normal distribution $\mathcal{N}(0, tI_d)$.

So, let us now prove that the random array $X(t)$, for $t < 1$, has normal distribution $\mathcal{N}(0, tI_d)$.

For this let $\varphi(t, \cdot)$ be the characteristic funtion of $X(t)$, where

$$\varphi(t, \tilde{u}) = \varphi(t; u_1, u_2, \cdots, u_d),\ \tilde{u} = (u_1, u_2, \cdots, u_d)^t \in \mathbb{R}^d,$$

and

$$\varphi(t, \tilde{u}) := E\left(e^{i\tilde{u}^t X(t)}\right).$$

We are to show that $\varphi(t, \tilde{u})$ satisfies the differential equation

$$\frac{\partial \varphi}{\partial t}(t, \tilde{u}) = -\frac{1}{2}\sum_{l=1}^{d} u_l^2 \varphi(t, \tilde{u}),\ 0 \leqslant t < 1,\ \tilde{u} \in \mathbb{R}^d.$$

And since the system

$$\begin{array}{rcl} \varphi(t, \tilde{u}) & = & E\left(e^{i\tilde{u}^t X(t)}\right) \\ \frac{\partial \varphi}{\partial t}(t, \tilde{u}) & = & -\frac{1}{2}\sum_{l=1}^{d} u_l^2 \varphi(t, \tilde{u}) \end{array},\quad 0 \leqslant t < 1,\ \tilde{u} \in \mathbb{R}^d$$

has the unique solution (as we shall see below in the note, see also p. 155 in Billingsley [1968])

$$\varphi(t, \tilde{u}) = e^{-\frac{1}{2}\sum_{l=1}^{d} u_l^2 t}$$
$$\Longleftrightarrow$$
$$\varphi(t, \tilde{u}) = e^{-\frac{1}{2}\tilde{u}^t t I_d \tilde{u}},$$

this will lead to the normality of the distribution of $X(t)$, for $t<1$.

If f satisfies $f'(t) = A(t)f(t)$ with A continuous, then its ratio with the nonvanishing function $f_0 = \exp\int_0^t A(\tau)d\tau$ has derivative 0, so that $f(t)/f_0(t) = f(0)/f_0(0) = f(0)$.

So, it remains to prove that $\varphi(t,\tilde{u})$ satisfies the mentioned differential equation.
We want now to show that

$$\lim_{h\downarrow 0}\frac{\varphi(t+h,\tilde{u})-\varphi(t,\tilde{u})}{h} = -\frac{1}{2}\sum_{l=1}^{d} u_l^2\varphi(t,\tilde{u}),\ 0\leqslant t<1,\ \tilde{u}\in\mathbb{R}^d.$$

Where the left side of this equality is equal to $\frac{\partial\varphi}{\partial t}(t,\tilde{u})$ for $0\leqslant t<1,\ \tilde{u}\in\mathbb{R}^d$, since the right side is continuous in t.
Fix $t<1$ temporarily. For notational convenience let us put for $h>0$

$$\triangle(h) := \left(\triangle^1(h),\cdots,\triangle^d(h)\right)^t,$$

and

$$\triangle^l(h) := X^l(t+h)-X^l(t),\ l=1,2,\cdots,d.$$

From the assumptions we have

$$E(\triangle(h))=0,\ \text{and}\ E\left(\triangle(h)(\triangle(h))^t\right)=O(h),$$

where

$$O(h) := \begin{pmatrix} h & 0 & \cdots & 0\\ 0 & h & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & h\end{pmatrix}_{d\times d}.$$

$\varphi(t+h,\tilde{u})-\varphi(t,\tilde{u}) =$
$= E\left(e^{i\tilde{u}^tX(t)}\left(e^{i\tilde{u}^t\triangle(h)}-1\right)\right)$
$= E\left(e^{i\tilde{u}^tX(t)}\left(i\tilde{u}^t\triangle(h)-\frac{1}{2}\left(\tilde{u}^t\triangle(h)\right)^2+c(\tilde{u}^t\triangle(h))\right)\right).$
Where the function $c(\cdot)$ is satisfying that

$$e^{iv} = 1+iv-\frac{1}{2}v^2+c(v),\ \forall v\in\mathbb{R},$$

and $|c(v)|\leqslant\min(|v|^3,v^2)\ \forall v\in\mathbb{R}$.
For this, we see
$|c'''(v)| = |-ie^{iv}| = 1,\ \forall v\in\mathbb{R}$

$\Longrightarrow \left| \int\limits_0^v |c'''(\tau)|\, d\tau \right| = |v|$
$\Longrightarrow \left| \int\limits_0^v c'''(\tau) d\tau \right| \leqslant |v|$
$\Longrightarrow |c''(v)| \leqslant |v|$
$\Longrightarrow |c'(v)| \leqslant \frac{1}{2}v^2$
$\Longrightarrow |c(v)| \leqslant \frac{1}{6}|v|^3 \leqslant |v|^3, \ \forall v \in \mathbb{R},$
and also similarly,
$|c''(v)| = |-e^{iv} + 1| \leqslant 2$
$\Longrightarrow |c(v)| \leqslant v^2, \ \forall v \in \mathbb{R}.$
So, we conclude the validity of the inequality $|c(v)| \leqslant \min(|v|^3, v^2), \ \forall v \in \mathbb{R}$.
Now, since $X(t)$ and $\triangle(h)$ are independent, we have
$\varphi(t+h, \tilde{u}) - \varphi(t, \tilde{u}) =$
$= \varphi(t, \tilde{u}) \left(-\frac{1}{2} \sum\limits_{l=1}^{d} u_l^2 h + E\left(c\left(\tilde{u}^t \triangle (h)\right)\right) \right).$
But since $|\varphi(t, \tilde{u})| \leqslant 1$ we conclude

$$\left| \frac{\varphi(t+h, \tilde{u}) - \varphi(t, \tilde{u})}{h} + \frac{1}{2} \sum_{l=1}^{d} u_l^2 \varphi(t, \tilde{u}) \right| \leqslant \frac{1}{h} E\left(\left| c\left(\tilde{u}^t \triangle (h)\right) \right| \right),$$

and

$$\left| c\left(\tilde{u}^t \triangle (h)\right) \right| \leqslant \left| \tilde{u}^t \triangle (h) \right|^3.$$

Hence, it is sufficient to prove

$$\lim_{h \downarrow 0} \frac{1}{h} E\left(\left| \tilde{u}^t \triangle (h) \right|^3 \right) = 0.$$

But also we can assure that (by applying Hölder-inequality)

$$\left| \tilde{u}^t \triangle (h) \right|^3 \leqslant K \sum_{l=1}^{d} \left| u_l \triangle^l(h) \right|^3$$

is valid for some universal constant K, where the value of this constant depends only on d.
Therefore, it is sufficient to prove

$$\lim_{h \downarrow 0} \frac{1}{h} E\left(\left| \triangle^l(h) \right|^3 \right) = 0, \ \text{for } l = 1, 2, \cdots, d.$$

Since X^l, for each $l = 1, 2, \cdots, d$ satisfies all assumptions of theorem 19.1 in Billingsley [1968], and from the computations in p. 156 in the same book, we find that the wanted limit is valid for each $l = 1, 2, \cdots, d$.

Hence, the proof is complete and the assertion holds. $\square$

Definition 2.2.3. Let X_n, $n = 1, 2, \cdots$, be random elements defined on $(\Omega, \mathcal{A}, P)$ and taking values $(D[0,1])^d$. We say that the sequence $(X_n)_{n\in\mathbb{N}}$ has asymptotically independent increments iff for all partitions of $[0,1]$

$$0 \leqslant s_1 \leqslant t_1 < s_2 \leqslant t_2 < \cdots < s_k \leqslant t_k \leqslant 1, \; k \in \mathbb{N}.$$

Then the quantity

$$P\left\{X_n(t_j) - X_n(s_j) \in H_j, \; j = 1, 2, \cdots, k\right\} - \prod_{j=1}^{k} P\left\{X_n(t_j) - X_n(s_j) \in H_j\right\}$$

converges to zero as n tends to infinity, for all Borel sets $H_1, H_2, \cdots, H_k \in \mathbb{B}^d$, $k \in \mathbb{N}$.

Theorem 2.2.4. Let X_n, $n = 1, 2, \cdots$, be random elements defined on $(\Omega, \mathcal{A}, P)$ and taking values $(D[0,1])^d$. Suppose that the sequence $(X_n)_{n\in\mathbb{N}}$ has asymptotically independent increments, and the sequences $\left(\left(X_n^l(t)\right)^2\right)_{n\in\mathbb{N}}$, for $l = 1, 2, \cdots, d$, $t \in [0,1]$ are uniformly integrable. Suppose further that

$$E\left(X_n(t)\right) \underset{n \longrightarrow \infty}{\longrightarrow} 0,$$

and

$$E\left(X_n(t)\left(X_n(t)\right)^t\right) \underset{n \longrightarrow \infty}{\longrightarrow} tI_d, \; \forall t \in [0,1].$$

Suppose finally that, for each positive ε, and η, there exists a positive δ such that

$$P\left\{w(X_n, \delta) \geqslant \varepsilon\right\} \leqslant \eta, \; \forall n \geqslant n_0$$

is valid for some $n_0 \in \mathbb{N}$ large enough. Where

$$w(x, \delta) := \sup_{|s-t|<\delta} |x(s) - x(t)|, \; x = (x^1, \cdots, x^d)^t \in \left(D[0,1]\right)^d,$$

$0 < \delta \leqslant 1$, $s, t \in [0,1]$, and $|x(s) - x(t)| := \max\limits_{1\leqslant l\leqslant d} |x^l(s) - x^l(t)|$.

Then $X_n \xrightarrow{\mathcal{D}} W$.

Proof: Similarly to the proof of theorem 19.2 in Billingsley [1968], we can write here the following arguments:

Since $E\left(\left(X_n^l(0)\right)^2\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0$, for each $l = 1, 2, \cdots, d$, we conclude that the sequences $\left(X_n^l(0)\right)_{n\in\mathbb{N}}$, $l = 1, 2, \cdots, d$, are tight. And from theorem 15.5 in Billingsley [1968], we find that the sequences $\left(X_n^l\right)_{n\in\mathbb{N}}$, $l = 1, 2, \cdots, d$, are tight, also the sequence $(X_n)_{n\in\mathbb{N}}$ is tight. This means (by Prohorov's theorem) that each subsequence of $(X_n)_{n\in\mathbb{N}}$, contains further sub-subsequence that is convergent in distribution. Let X be the limit in distribution of a such sub-subsequence, then also from the same mentioned theorem 15.5 we find that $P\left\{X^l \in C[0,1]\right\} = 1$, $l = 1, \cdots, d$. This implies $P\left\{X \in (C[0,1])^d\right\} = 1$. We are to show that any such X must be distributed as W.
First, without any loss of generality we can assume that

$$X_n \xrightarrow{\mathcal{D}} X.$$

From the assumptions, and from theorem 5.4 in Billingsley, we find that

$$E\left(X^l(t)\right) = 0,\ l = 1, \cdots, d,\ \forall t \in [0,1],$$

and

$$E\left(X^{l_1}(t)X^{l_2}(t)\right) = \begin{cases} 0 & : \ l_1 \neq l_2 \\ t & : \ l_1 = l_2 \end{cases},\ l_1, l_2 = 1, \cdots, d,\ \forall t \in [0,1].$$

These equalities imply that

$$E\left(X(t)\right) = 0,\ E\left(X(t)\left(X(t)\right)^t\right) = tI_d,\ \forall t \in [0,1].$$

Now, since $(X_n)_{n\in\mathbb{N}}$ has asymptotically independent increments, we have for all partitions of $[0,1]$

$$0 \leqslant s_1 \leqslant t_1 < s_2 \leqslant t_2 < \cdots < s_k \leqslant t_k \leqslant 1,\ k \in \mathbb{N},$$

the quantity

$$P\left\{X_n(t_j) - X_n(s_j) \in H_j,\ j = 1, 2, \cdots, k\right\} - \prod_{j=1}^{k} P\left\{X_n(t_j) - X_n(s_j) \in H_j\right\}$$

converges to zero as n tends to infinity, for all Borel sets $H_1, H_2, \cdots, H_k \in \mathbb{B}^d$, $k \in \mathbb{N}$.
Since $X_n \xrightarrow{\mathcal{D}} X$ we have

$$\begin{pmatrix} X_n(t_1) - X_n(s_1) \\ \vdots \\ X_n(t_k) - X_n(s_k) \end{pmatrix} \xrightarrow{\mathcal{D}} \begin{pmatrix} X(t_1) - X(s_1) \\ \vdots \\ X(t_k) - X(s_k) \end{pmatrix}.$$

And consequently, we find

$$P\left\{X(t_j)-X(s_j)\in H_j,\ j=1,2,\cdots,k\right\}-\prod_{j=1}^{k}P\left\{X(t_j)-X(s_j)\in H_j\right\}=0,$$

for all Borel sets $H_1,\ H_2,\ \cdots,\ H_k\in\mathbb{B}^d$, $k\in\mathbb{N}$, satisfying that

$$P\left\{X(t_j)-X(s_j)\in\partial H_j\right\}=0,\ j=1,2,\cdots,k.$$

Where for each j, the set ∂H_j stands for the boundary of the set H_j. This will lead to the validity of the equality

$$P\left\{X(t_j)-X(s_j)\in H_j,\ j=1,2,\cdots,k\right\}=\prod_{j=1}^{k}P\left\{X(t_j)-X(s_j)\in H_j\right\},$$

for all Borel sets $H_1,\ H_2,\ \cdots,\ H_k\in\mathbb{B}^d$, $k\in\mathbb{N}$.
This means that the increments $X(t_j)-X(s_j)$, for $j=1,2,\cdots,k$, are independent under P.
Let s_{j+1} tends to t_j, for $j=1,\cdots,k-1$. Then

$$X(t_{j+1})-X(s_{j+1})\underset{s_{j+1}\longrightarrow t_j}{\longrightarrow}X(t_{j+1})-X(t_j)\ [P],\ \text{for } j=1,\cdots,k-1.$$

Hence,

$$\begin{pmatrix}X(t_1)-X(s_1)\\X(t_2)-X(s_2)\\X(t_3)-X(s_3)\\\vdots\\X(t_k)-X(s_k)\end{pmatrix}\xrightarrow{\mathcal{D}}\begin{pmatrix}X(t_1)-X(s_1)\\X(t_2)-X(t_1)\\X(t_3)-X(t_2)\\\vdots\\X(t_k)-X(t_{k-1})\end{pmatrix}.$$

Put $t_0:=s_1$ we conclude easily

$$P\left\{X(t_j)-X(t_{j-1})\in H_j,\ j=1,2,\cdots,k\right\}=\prod_{j=1}^{k}P\left\{X(t_j)-X(t_{j-1})\in H_j\right\},$$

for all partitions of $[0,1]$, $0\leqslant t_0\leqslant t_1\leqslant\cdots\leqslant t_k\leqslant 1$, and for all Borel sets $H_1,\ H_2,\ \cdots,\ H_k\in\mathbb{B}^d$, $k\in\mathbb{N}$, satisfying that

$$P\left\{X(t_j)-X(s_j)\in\partial H_j\right\}=0,\ j=1,2,\cdots,k.$$

And also this will imply that

$$P\left\{X(t_j)-X(t_{j-1})\in H_j,\ j=1,2,\cdots,k\right\}=\prod_{j=1}^{k}P\left\{X(t_j)-X(t_{j-1})\in H_j\right\},$$

for all partitions of $[0,1]$, $0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant 1$, and for all Borel sets $H_1, H_2, \cdots, H_k \in \mathbb{B}^d$, $k \in \mathbb{N}$.
This means that X has independent increments by definition indeed.
And therefore, the assertion $X \sim W$ is valid by theorem 2.2.2. □

Theorem 2.2.5. Let X be a random element defined on $(\Omega, \mathcal{A}, P)$ and taking values in $(D[0,1])^d$, such that

$$P\{X \in (C[0,1])^d\} = 1, \ P\{X(0) = 0\} = 1.$$

Suppose that the functions ρ, σ^2 are defined and continuous on $[0,1)$ and taking values in $\mathbb{R}^d$.

$$\rho := (\rho_1, \cdots, \rho_d)^t, \ \sigma^2 := \left(\sigma_1^2, \cdots, \sigma_d^2\right)^t,$$

and for each l, the function σ_l^2 is nonnegative. Let us put for $t \in [0,1)$

$$g_l(t) := \exp\left(\int_0^t \rho_l(\tau) d\tau\right), \ l = 1, 2, \cdots, d,$$

$$G_l(t) := \int_0^t \sigma_l^2(r) g_l^{-2}(r) dr, \ l = 1, 2, \cdots, d.$$

Suppose that the limits $\lim\limits_{t \longrightarrow 1} g_l(t) G_l(t)$ exist and are finite.
Suppose finally that X satisfies the conditions [1] (or [1a]), [2] (or [2a]), and [3] (or [3a]), where

[**1**] If $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k < 1$, then

[**I**] $$\lim_{h \downarrow 0} \frac{1}{h} E\left\{\left| E\left\{X^l(t_k + h) - X^l(t_k) \middle| X(t_1), X(t_2), \cdots, X(t_k)\right\} - \right.\right.$$
$$\left.\left. - h\rho_l(t_k) X^l(t_k)\right|\right\} = 0, \text{ for } l = 1, \cdots, d,$$

[**II**] $$\lim_{h \downarrow 0} \frac{1}{h} E\left\{\left| E\left\{\left(X^l(t_k + h) - X^l(t_k)\right)^2 \middle| X(t_1), X(t_2), \cdots, X(t_k)\right\} - \right.\right.$$
$$\left.\left. - h\sigma_l^2(t_k)\right|\right\} = 0, \text{ for } l = 1, \cdots, d,$$

and

[**III**] $\lim_{h\downarrow 0}\frac{1}{h}E\left\{\left|E\left\{\left(X^{l_1}(t_k+h)-X^{l_1}(t_k)\right)\left(X^{l_2}(t_k+h)-X^{l_2}(t_k)\right)\right|\right.\right.$

$$\left.\left.\left|X(t_1),X(t_2),\cdots,X(t_k)\right\}\right|\right\}=0,\text{ for } l_1\neq l_2.$$

[**1a**] If $0\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_k<1$, then for all $\tilde{u}_1,\ \tilde{u}_2,\ \cdots,\ \tilde{u}_k\in\mathbb{R}^d$

[**I**] $\lim_{h\downarrow 0}\frac{1}{h}E\left\{\left(\exp i\sum_{j=1}^{k}\tilde{u}_j^tX(t_j)\right)\left(X^l(t_k+h)-X^l(t_k)-\right.\right.$

$$\left.\left.-h\rho_l(t_k)X^l(t_k)\right)\right\}=0,\text{ for } l=1,\cdots,d,$$

[**II**] $\lim_{h\downarrow 0}\frac{1}{h}E\left\{\left(\exp i\sum_{j=1}^{k}\tilde{u}_j^tX(t_j)\right)\left(\left(X^l(t_k+h)-X^l(t_k)\right)^2-\right.\right.$

$$\left.\left.-h\sigma_l^2(t_k)\right)\right\}=0,\text{ for } l=1,\cdots,d,$$

and

[**III**] $\lim_{h\downarrow 0}\frac{1}{h}E\left\{\left(\exp i\sum_{j=1}^{k}\tilde{u}_j^tX(t_j)\right)\left(X^{l_1}(t_k+h)-X^{l_1}(t_k)\right)\right.$

$$\left.\left(X^{l_2}(t_k+h)-X^{l_2}(t_k)\right)\right\}=0,\text{ for } l_1\neq l_2.$$

[**2**] $\sup_{t\in[0,1]}E\left(\left(X^l(t)\right)^2\right)<\infty$, for $l=1,2,\cdots,d$.

[**2a**] $\lim_{\alpha\longrightarrow\infty}\sup_{t\in[0,1]}\int_{\{|X^l(t)|>\alpha\}}\left|X^l(t)\right|dP=0$, for $l=1,2,\cdots,d$.

[**3**] There is a constant K such that for each $l=1,\cdots,d$

$$E\left\{\left(X^l(t)-X^l(t_1)\right)^2\left(X^l(t_2)-X^l(t)\right)^2\right\}<K\left(t_2-t_1\right)^2,$$

for all t_1, t, t_2 satisfying $0\leqslant t_1\leqslant t\leqslant t_2\leqslant 1$.

[**3a**] $\lim_{\alpha\longrightarrow\infty}\limsup_{h\downarrow 0}\int_{\left\{\left(X^l(t+h)-X^l(t)\right)^2\geqslant\alpha h\right\}}\left(X^l(t+h)-X^l(t)\right)^2dP=0$,

for each $l=1,\cdots,d$, and for all $0\leqslant t<1$.

Then X is the continuous Gaussian function specified by

$$E(X(t)) = 0, \ \forall t \in [0,1],$$

and for $0 \leqslant s \leqslant t \leqslant 1$

$$E\left(X^{l_1}(s)X^{l_2}(t)\right) = \begin{cases} \int\limits_0^s \sigma_l^2(r)\exp\left(\int\limits_r^s \rho_l(\tau)d\tau + \int\limits_r^t \rho_l(\tau)d\tau\right)dr & , \text{for } l_1 = l_2 \\ 0 & , \text{for } l_1 \neq l_2 \end{cases}.$$

More specifically,

$$(X(t))_{0\leqslant t\leqslant 1} \sim \begin{pmatrix} g_1(t)\left(1+G_1(t)\right)W^{o1}_{\left(\frac{G_1(t)}{1+G_1(t)}\right)} \\ \vdots \\ g_d(t)\left(1+G_d(t)\right)W^{od}_{\left(\frac{G_d(t)}{1+G_d(t)}\right)} \end{pmatrix}_{0\leqslant t\leqslant 1},$$

where here, the random variables W^{o1}, W^{o2}, $\cdots$, W^{od}, are independent one-dimensional Brownian Bridges, and also we put

$$\frac{G_l(1)}{1+G_l(1)} := \lim_{t\longrightarrow 1}\frac{G_l(t)}{1+G_l(t)}, \ l = 1,2,\cdots,d,$$

and

$$g_l(1)\left(1+G_l(1)\right) := \lim_{t\longrightarrow 1} g_l(t)\left(1+G_l(t)\right), \ l = 1,2,\cdots,d.$$

Proof: In the proof we assume that the conditions (1a), (2a), and (3a) are satisfied. Where similarly to theorem 19.3 in Billingsley [1968], we can assure that (1)$\Longrightarrow$(1a), (2)$\Longrightarrow$(2a), and (3)$\Longrightarrow$(3a).
We shall show first that all finite-dimensional distributions of X are normal. Secondly, we show that X satisfies the mentioned assumptions, related with the expectations $E(X(t))$, for $0 \leqslant t \leqslant 1$, and also $E\left(X^{l_1}(s)X^{l_2}(t)\right)$, for $0 \leqslant s \leqslant t \leqslant 1$, $1 \leqslant l_1, l_2 \leqslant d$. Finally, we clarify the mentioned form of the distribution of X.

For $k \in \mathbb{N}$, let us fix points t_j, $j = 1,\cdots,k$, such that $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k < 1$, also we fix $\tilde{u}_1$, $\tilde{u}_2$, $\cdots$, $\tilde{u}_k \in \mathbb{R}^d$. Let t, $\tilde{u}$ vary over strip $t_k \leqslant t < 1$, $\tilde{u} \in \mathbb{R}^d$.
Put

$$\psi(t,\tilde{u}) := E\left\{\exp\left(i\left[\tilde{u}_1^t X(t_1) + \cdots + \tilde{u}_k^t X(t_k) + \tilde{u}^t X(t)\right]\right)\right\},$$

and

$$Z := \tilde{u}_1^t X(t_1) + \cdots + \tilde{u}_k^t X(t_k).$$

So, we can write

$$\psi(t, \tilde{u}) = E\left(e^{iZ} e^{i\tilde{u}^t X(t)}\right).$$

For notational convenience, let us put also

$$\tilde{u}_j := (u_{1j}, \cdots, u_{dj})^t, \text{ for } j = 1, \cdots, k, \text{ and } \tilde{u} := (u_1, \cdots, u_d)^t.$$

We are to show that $\psi(t, \tilde{u})$ satisfies the differential equation

$$\frac{\partial \psi}{\partial t}(t, \tilde{u}) = \sum_{l=1}^{d} \left(u_l \rho_l(t) \frac{\partial \psi}{\partial u_l}(t, \tilde{u}) - \frac{1}{2} u_l^2 \sigma_l^2(t) \psi(t, \tilde{u}) \right),$$

and next we shall solve this equation.
To do this, we prove the validity of the limit

$$\lim_{h \downarrow 0} \left| \tfrac{1}{h} \left[\psi(t+h, \tilde{u}) - \psi(t, \tilde{u}) \right] - \sum_{l=1}^{d} \left(u_l \rho_l(t) \tfrac{\partial \psi}{\partial u_l}(t, \tilde{u}) - \tfrac{1}{2} u_l^2 \sigma_l^2(t) \psi(t, \tilde{u}) \right) \right| = 0.$$

And we prove also that the function

$$\sum_{l=1}^{d} \left(u_l \rho_l(t) \frac{\partial \psi}{\partial u_l}(t, \tilde{u}) - \frac{1}{2} u_l^2 \sigma_l^2(t) \psi(t, \tilde{u}) \right)$$

is continuous in t, where this will lead to the validity of the equality

$$\frac{\partial \psi}{\partial t}(t, \tilde{u}) = \lim_{h \downarrow 0} \frac{1}{h} \left[\psi(t+h, \tilde{u}) - \psi(t, \tilde{u}) \right].$$

For this, we deal with the functions ψ, $\frac{\partial \psi}{\partial u_l}$, $l = 1, \cdots, d$.
We begin with $\psi(t, \tilde{u}) = E\left(e^{iZ} e^{i\tilde{u}^t X(t)}\right)$
It is clear that the function ψ is continuous in $t, \tilde{u}$ jointly in the strip

$$t_k \leqslant t < 1, \; \tilde{u} \in \mathbb{R}^d,$$

since

$$\left| e^{iZ} e^{i\tilde{u}^t X(t)} \right| = 1, \; P\left\{ X \in (C[0,1])^d \right\} = 1,$$

and if $(t_n, \tilde{u}_n) \underset{n \longrightarrow \infty}{\longrightarrow} (t, \tilde{u})$, then

$$\left(e^{iZ} e^{i\tilde{u}_n^t X(t_n)}\right) \underset{n \longrightarrow \infty}{\longrightarrow} \left(e^{iZ} e^{i\tilde{u}^t X(t)}\right) \; [P].$$

And consequently, by the bounded convergence theorem we have

$$E\left(e^{iZ} e^{i\tilde{u}_n^t X(t_n)}\right) \underset{n \longrightarrow \infty}{\longrightarrow} E\left(e^{iZ} e^{i\tilde{u}^t X(t)}\right).$$

Also, for each $l = 1, \cdots, d$ the equality

$$\frac{\partial \psi}{\partial u_l}(t, \tilde{u}) = E\left(iX^l(t)e^{iZ}e^{i\tilde{u}^t X(t)}\right)$$

is valid, since

$$\left|iX^l(t)e^{iZ}e^{i\tilde{u}^t X(t)}\right| = \left|X^l(t)\right|,$$

and from the condition (2a) (also see the arguments in p. 32 in Billingsley [1968]).
Moreover, the condition (2a) implies also that the function $\frac{\partial \psi}{\partial u_l}$ is continuous in $t, \tilde{u}$ jointly in the strip

$$t_k \leqslant t < 1, \;\; \tilde{u} \in \mathbb{R}^d.$$

To explain this we put

$$\Lambda(t, \tilde{u}) := iX^l(t)e^{iZ}e^{i\tilde{u}^t X(t)}.$$

So, we can write

$$\frac{\partial \psi}{\partial u_l}(t, \tilde{u}) = E(\Lambda(t, \tilde{u})).$$

If $(t_n, \tilde{u}_n) \underset{n \longrightarrow \infty}{\longrightarrow} (t, \tilde{u})$, then $E(\Lambda(t_n, \tilde{u}_n)) \underset{n \longrightarrow \infty}{\longrightarrow} E(\Lambda(t, \tilde{u}))$,
since the sequence $(\Lambda(t_n, \tilde{u}_n))_{n \in \mathbb{N}}$ is uniformly integrable, and also

$$\Lambda(t_n, \tilde{u}_n) \underset{n \longrightarrow \infty}{\longrightarrow} \Lambda(t, \tilde{u}) \;\; [P],$$

(see theorem 5.4 in Billingsley [1968]).
So far, we have proved that the functions ψ, $\frac{\partial \psi}{\partial u_l}$, for $l = 1, \cdots, d$, are continuous in the strip

$$t_k \leqslant t < 1, \;\; \tilde{u} \in \mathbb{R}^d.$$

We turn now to prove the validity of the limit

$$\lim_{h \downarrow 0} \left| \frac{1}{h}\left[\psi(t+h, \tilde{u}) - \psi(t, \tilde{u})\right] - \sum_{l=1}^{d}\left(u_l \rho_l(t) \frac{\partial \psi}{\partial u_l}(t, \tilde{u}) - \frac{1}{2}u_l^2 \sigma_l^2(t)\psi(t, \tilde{u})\right)\right| = 0.$$

in the same strip.
To simplify the computations we put for $h > 0$

$$\triangle(h) := \left(\triangle^1(h), \cdots, \triangle^d(h)\right)^t,$$

and

$$\triangle^l(h) := X^l(t+h) - X^l(t), \;\; l = 1, 2, \cdots, d.$$

$\psi(t+h, \tilde{u}) - \psi(t, \tilde{u}) =$
$= E\left(e^{iZ}e^{i\tilde{u}^t X(t)}\left(e^{i\tilde{u}^t \triangle(h)} - 1\right)\right)$

$= E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\left(i\tilde{u}^t \triangle (h) - \frac{1}{2}\left(\tilde{u}^t \triangle (h)\right)^2 + c(\tilde{u}^t \triangle (h))\right)\right).$

Where the function $c(\cdot)$ is satisfying that

$$e^{iv} = 1 + iv - \frac{1}{2}v^2 + c(v),\ \forall v \in \mathbb{R},$$

and $|c(v)| \leqslant \min(|v|^3, |v|^2)\ \forall v \in \mathbb{R}$.

$\left|\frac{1}{h}\left[\psi(t+h,\tilde{u}) - \psi(t,\tilde{u})\right] - \sum_{l=1}^{d}\left(u_l\rho_l(t)\frac{\partial\psi}{\partial u_l}(t,\tilde{u}) - \frac{1}{2}u_l^2\sigma_l^2(t)\psi(t,\tilde{u})\right)\right| \leqslant$

$\leqslant \left|\frac{1}{h}E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\left(i\tilde{u}^t \triangle (h)\right)\right) - \sum_{l=1}^{d}u_l\rho_l(t)\frac{\partial\psi}{\partial u_l}(t,\tilde{u})\right| +$

$+\left|\frac{1}{h}E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\left(-\frac{1}{2}\left(\tilde{u}^t \triangle (h)\right)^2\right)\right) + \frac{1}{2}\sum_{l=1}^{d}u_l^2\sigma_l^2(t)\psi(t,\tilde{u})\right| +$

$+\left|\frac{1}{h}E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\left(c(\tilde{u}^t \triangle (h))\right)\right)\right|$

$\leqslant \sum_{l=1}^{d}\frac{|u_l|}{h}\left|E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\left[\triangle^l(h) - h\rho_l(t)X^l(t)\right]\right)\right| +$

$+\sum_{l=1}^{d}\frac{u_l^2}{2h}\left|E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\left[\left(\triangle^l(h)\right)^2 - h\sigma_l^2(t)\right]\right)\right| +$

$+\sum_{l_1\neq l_2}\frac{\left|u_{l_1}u_{l_2}\right|}{2h}\left|E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\left[\triangle^{l_1}(h)\triangle^{l_2}(h)\right]\right)\right| +$

$+\frac{1}{h}E\left(|c(\tilde{u}^t \triangle (h))|\right).$

But from condition (1a) we conclude that the quantity

$\sum_{l=1}^{d}\frac{|u_l|}{h}\left|E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\left[\triangle^l(h) - h\rho_l(t)X^l(t)\right]\right)\right| +$

$+\sum_{l=1}^{d}\frac{u_l^2}{2h}\left|E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\left[\left(\triangle^l(h)\right)^2 - h\sigma_l^2(t)\right]\right)\right| +$

$+\sum_{l_1\neq l_2}\frac{\left|u_{l_1}u_{l_2}\right|}{2h}\left|E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\left[\triangle^{l_1}(h)\triangle^{l_2}(h)\right]\right)\right|$

tends to zero as $h \downarrow 0$. It remains to show

$$\frac{1}{h}E\left(\left|c(\tilde{u}^t \triangle (h))\right|\right) \underset{h\downarrow 0}{\longrightarrow} 0.$$

Since $|c(v)| \leqslant \min(|v|^3, |v|^2)\ \forall v \in \mathbb{R}$, we can assure that

$$\frac{1}{h}E\left(|c(\tilde{u}^t \triangle (h))|\right) \leqslant K\left(\max_{1\leqslant l\leqslant d}\left(u_l^2\right)\frac{1}{h}\sum_{l=1}^{d}\int_{\left(\triangle^l(h)\right)^2\geqslant\alpha h}\left(\triangle^l(h)\right)^2 dP + \right.$$
$$\left. + \max_{1\leqslant l\leqslant d}\left(|u_l|^3\right)\frac{1}{h}\sum_{l=1}^{d}\int_{\left(\triangle^l(h)\right)^2<\alpha h}\left|\triangle^l(h)\right|^3 dP\right)$$

is valid for some universal constant K, where the value of this constant depends only on d.

For example take $d=2$, then
$E\left(|c(\tilde{u}^t \triangle (h))|\right) = E\left(\left|c\left(u_1 \triangle^1 (h) + u_2 \triangle^2 (h)\right)\right|\right)$

$$= \int\limits_{\substack{\left(\triangle^1(h)\right)^2 \geqslant \alpha h \\ \left(\triangle^2(h)\right)^2 \geqslant \alpha h}} + \int\limits_{\substack{\left(\triangle^1(h)\right)^2 \geqslant \alpha h \\ \left(\triangle^2(h)\right)^2 < \alpha h}} + \int\limits_{\substack{\left(\triangle^1(h)\right)^2 < \alpha h \\ \left(\triangle^2(h)\right)^2 \geqslant \alpha h}} + \int\limits_{\substack{\left(\triangle^1(h)\right)^2 < \alpha h \\ \left(\triangle^2(h)\right)^2 < \alpha h}} .$$

Now, for the first, second and the third integral we use the inequality $|c(v)| \leqslant (|v|^2)$, but for the fourth one we use $|c(v)| \leqslant (|v|^3)$. Also we use $(v_1+v_2)^2 \leqslant K(v_1^2+v_2^2)$, $|v_1+v_2|^3 \leqslant K(|v_1|^3+|v_2|^3)$. Finally, inside the second and the third integrals, we replace $\left(\triangle^1(h)\right)^2$, and $\left(\triangle^2(h)\right)^2$ by their maximum, and we use the linear property for all integrals.

$$\frac{1}{h}E\left(|c(\tilde{u}^t \triangle (h))|\right) \leqslant K\left(\max_{1\leqslant l\leqslant d}\left(u_l^2\right)\frac{1}{h}\sum_{l=1}^{d}\int\limits_{\left(\triangle^l(h)\right)^2 \geqslant \alpha h}\left(\triangle^l(h)\right)^2 dP\right) + \\ +K\max_{1\leqslant l\leqslant d}\left(|u_l|^3\right)\alpha^{\frac{3}{2}}h^{\frac{1}{2}}d$$

Hence, by using the condition (3a) we find

$$\lim_{h\downarrow 0}\frac{1}{h}E\left(\left|c(\tilde{u}^t \triangle (h))\right|\right) = 0.$$

So far, we have proved that the function $\psi(t,\tilde{u}) = E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\right)$ satisfies the differential equation

$$\frac{\partial\psi}{\partial t}(t,\tilde{u}) = \sum_{l=1}^{d}\left(u_l\rho_l(t)\frac{\partial\psi}{\partial u_l}(t,\tilde{u}) - \frac{1}{2}u_l^2\sigma_l^2(t)\psi(t,\tilde{u})\right)$$

in the strip

$$t_k \leqslant t < 1,\ \tilde{u} \in \mathbb{R}^d.$$

Now, we shall solve this equation and find the general form $\psi(t,\tilde{u})$ that satisfies this equation together with

$$\psi(t,\tilde{u}) = E\left(e^{iZ}e^{i\tilde{u}^tX(t)}\right).$$

For this, let us put for arbitrary $\tilde{v} = (v_1,\cdots,v_d)^t \in \mathbb{R}^d$

$$\lambda_{v_l}(s) := v_l\exp\left(-\int\limits_{t_k}^{s}\rho_l(\tau)d\tau\right),\ l=1,2,\cdots,d,\ \text{and}\ t_k \leqslant s < 1,$$

$$\lambda_{\tilde{v}} := \left(\lambda_{v_1},\cdots,\lambda_{v_d}\right)^t,$$

and

$$g_{\tilde{v}}(s) := \psi\left(s, \lambda_{\tilde{v}}(s)\right),\ t_k \leqslant s < 1.$$

Let us compute the derivative $\frac{d}{ds} g_{\tilde{v}}(s)$. By the chain rule we have

$$g'_{\tilde{v}}(s) = \psi_1\left(s, \lambda_{\tilde{v}}(s)\right) + \sum_{l=1}^{d} \psi_{2l}\left(s, \lambda_{\tilde{v}}(s)\right) \lambda'_{vl}(s),$$

$$g'_{\tilde{v}}(s) = \psi_1\left(s, \lambda_{\tilde{v}}(s)\right) + \sum_{l=1}^{d} \psi_{2l}\left(s, \lambda_{\tilde{v}}(s)\right) \left[-\rho_l(s)\lambda_{vl}(s)\right],$$

where

$$\psi_1\left(t, \tilde{u}\right) := \frac{\partial \psi}{\partial t}\left(t, \tilde{u}\right),$$

and

$$\psi_{2l}\left(t, \tilde{u}\right) := \frac{\partial \psi}{\partial u_l}\left(t, \tilde{u}\right),\ l = 1, \cdots, d.$$

But also we have have

$$\psi_1\left(t, \tilde{u}\right) = \sum_{l=1}^{d} \left(u_l \rho_l(t) \psi_{2l}\left(t, \tilde{u}\right) - \frac{1}{2} u_l^2 \sigma_l^2(t) \psi\left(t, \tilde{u}\right)\right),$$

$$\psi_1\left(s, \lambda_{\tilde{v}}(s)\right) = \sum_{l=1}^{d} \left(\lambda_{v_l}(s)\rho_l(s)\psi_{2l}\left(s, \lambda_{\tilde{v}}(s)\right) - \frac{1}{2}\lambda_{v_l}^2(s)\sigma_l^2(s)\psi\left(s, \lambda_{\tilde{v}}(s)\right)\right),$$

and

$$\psi_1\left(s, \lambda_{\tilde{v}}(s)\right) = \sum_{l=1}^{d} \left(\lambda_{v_l}(s)\rho_l(s)\psi_{2l}\left(s, \lambda_{\tilde{v}}(s)\right) - \frac{1}{2}\lambda_{v_l}^2(s)\sigma_l^2(s) g_{\tilde{v}}(s)\right).$$

Hence,

$$g'_{\tilde{v}}(s) = \left(-\frac{1}{2}\sum_{l=1}^{d} \lambda_{v_l}^2(s)\sigma_l^2(s)\right) g_{\tilde{v}}(s).$$

Consequently, we can write

$$\frac{g'_{\tilde{v}}(s)}{g_{\tilde{v}}(s)} = -\frac{1}{2}\sum_{l=1}^{d} \lambda_{v_l}^2(s)\sigma_l^2(s)$$

$$\int_{t_k}^{s} \frac{g'_{\tilde{v}}(r)}{g_{\tilde{v}}(r)} dr = -\frac{1}{2}\sum_{l=1}^{d} \int_{t_k}^{s} \lambda_{v_l}^2(r)\sigma_l^2(r) dr$$

$$g_{\tilde{v}}(s) = g_{\tilde{v}}\left(t_k\right)\exp\left(-\frac{1}{2}\sum_{l=1}^{d}\int\limits_{t_k}^{s}\lambda_{v_l}^2(r)\sigma_l^2(r)dr\right)$$

$$\psi\left(s, \lambda_{\tilde{v}}(s)\right) = \psi\left(t_k, \lambda_{\tilde{v}}(t_k)\right)\exp\left(-\frac{1}{2}\sum_{l=1}^{d}\int\limits_{t_k}^{s}\lambda_{v_l}^2(r)\sigma_l^2(r)dr\right)$$

$$\psi\left(s, \lambda_{\tilde{v}}(s)\right) = \psi\left(t_k, \tilde{v}\right)\exp\left(-\frac{1}{2}\sum_{l=1}^{d}\int\limits_{t_k}^{s}\lambda_{v_l}^2(r)\sigma_l^2(r)dr\right).$$

Let $(t, \tilde{u})$ be an arbitrary point in the strip

$$t_k \leqslant t < 1,\ \tilde{u} \in \mathbb{R}^d,$$

and let us take

$$v_l = u_l \exp\left(\int\limits_{t_k}^{t}\rho_l(\tau)d\tau\right),\ \text{for } l = 1, \cdots, d.$$

Then we obtain

$$\lambda_{\tilde{v}}(t) = \tilde{u},$$

$$\lambda_{v_l}^2(r) = u_l^2 \exp\left(2\int\limits_{r}^{t}\rho_l(\tau)d\tau\right),\ \text{for } l = 1, \cdots, d.$$

Therefore, we find that the equality

$$\psi\left(t, \tilde{u}\right) = \psi\left(t_k, \tilde{u} * a\right)\exp\left(-\frac{1}{2}\sum_{l=1}^{d}u_l^2 b_l^2\right),$$

is valid in the strip

$$t_k \leqslant t < 1,\ \tilde{u} \in \mathbb{R}^d.$$

Where

$$\tilde{u} := \left(u_1, \cdots, u_d\right)^t,$$

$$a := \left(a_1, \cdots, a_d\right)^t,$$

$$\tilde{u} * a := \left(u_1 a_1, \cdots, u_d a_d\right)^t,$$

$$a_l := \exp\left(\int\limits_{t_k}^{t}\rho_l(\tau)d\tau\right),\ l = 1, \cdots, d,$$

and

$$b_l^2 := \int\limits_{t_k}^{t} \sigma_l^2(r) \exp\left(2 \int\limits_{r}^{t} \rho_l(\tau) d\tau \right) dr, \ l = 1, \cdots, d.$$

Now, let us put

$$b^2 := \begin{pmatrix} b_1^2 & 0 & \cdots & 0 \\ 0 & b_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_d^2 \end{pmatrix}.$$

Consequently,

$$\psi(t, \tilde{u}) = \psi(t_k, \tilde{u} * a) \exp\left(-\frac{1}{2} \tilde{u}^t b^2 \tilde{u} \right),$$

and also by using

$$\psi(t, \tilde{u}) = E\left(e^{iZ} e^{i\tilde{u}^t X(t)} \right)$$

we get

$$E\left\{ \exp\left(i\left[\tilde{u}_1^t X(t_1) + \cdots + \tilde{u}_k^t X(t_k) + \tilde{u}^t X(t) \right] \right) \right\} =$$
$$= E\left\{ \exp\left(i\left[\tilde{u}_1^t X(t_1) + \cdots + \tilde{u}_k^t X(t_k) + (\tilde{u} * a)^t X(t_k) \right] \right) \right\} \exp\left(-\tfrac{1}{2} \tilde{u}^t b^2 \tilde{u} \right),$$

for

$$0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k \leqslant t < 1.$$

And because of continuity, it is valid also for all partitions of $[0,1]$

$$0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k \leqslant t \leqslant 1.$$

Let $\tilde{v} \in \mathbb{R}^d$ be arbitrary, then

$$E\left\{ \exp\left(i\left[\tilde{u}_1^t X(t_1) + \cdots + \tilde{u}_k^t X(t_k) + \tilde{v}^t \left(X(t) - a * X(t_k) \right) \right] \right) \right\} =$$
$$= E\left\{ \exp\left(i\left[\tilde{u}_1^t X(t_1) + \cdots + (\tilde{u}_k - \tilde{v} * a)^t X(t_k) + \tilde{v}^t X(t) \right] \right) \right\}$$
$$= E\left\{ \exp\left(i\left[\tilde{u}_1^t X(t_1) + \cdots + \tilde{u}_k^t X(t_k) \right] \right) \right\} \exp\left(-\tfrac{1}{2} \tilde{v}^t b^2 \tilde{v} \right).$$

Hence, we conclude that

$$E\left\{ \exp\left(i\left[\tilde{u}_1^t X(t_1) + \cdots + \tilde{u}_k^t X(t_k) \right] + i\tilde{v}^t \left(X(t) - a * X(t_k) \right) \right) \right\} =$$
$$= E\left\{ \exp\left(i\left[\tilde{u}_1^t X(t_1) + \cdots + \tilde{u}_k^t X(t_k) \right] \right) \right\} \exp\left(-\tfrac{1}{2} \tilde{v}^t b^2 \tilde{v} \right)$$

$$= E\left\{\exp\left(i\left[\tilde{u}_1^t X(t_1) + \cdots + \tilde{u}_k^t X(t_k)\right]\right)\right\}\cdot$$

$$\cdot E\left\{\exp\left(i\tilde{v}^t\left[X(t) - a * X(t_k)\right]\right)\right\}.$$

This means that the random arrays $[X(t) - a * X(t_k)]$ and $\Big(X(t_1), \cdots, X(t_k)\Big)$ are independent under P, also $[X(t) - a * X(t_k)]$ is distributed as $\mathcal{N}(0, b^2)$.
And since $P\{X(0) = 0\} = 1$, we conclude easily that the random arrays

$$[X(t_k) - a\,(t_{k-1}, t_k) * X(t_{k-1})], [X(t_{k-1}) - a\,(t_{k-2}, t_{k-1}) * X(t_{k-2})], \cdots$$
$$\cdots, [X(t_2) - a\,(t_1, t_2) * X(t_1)], [X(t_1) - a\,(t_0, t_1) * X(t_0)], X(t_0)$$

are independent and distributed normaly under P, for all partitions

$$0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant 1.$$

Where explicitly we can write

$$a_l\,(\alpha, \beta) = \exp\left(\int_\alpha^\beta \rho_l(\tau) d\tau\right),\ l = 1, \cdots, d, \text{ and } \alpha, \beta \in [0, 1].$$

Moreover,
Let us put

$$a^* := \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_d \end{pmatrix},$$

and for fixed points $0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant 1$, we put

$$A := \begin{pmatrix} I_d & 0 & 0 & 0 & \cdots \\ \prod_{j=1}^{1} a^*(t_{j-1}, t_j) & I_d & 0 & 0 & \cdots \\ \prod_{j=1}^{2} a^*(t_{j-1}, t_j) & \prod_{j=2}^{2} a^*(t_{j-1}, t_j) & I_d & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \prod_{j=1}^{k-1} a^*(t_{j-1}, t_j) & \prod_{j=2}^{k-1} a^*(t_{j-1}, t_j) & \prod_{j=3}^{k-1} a^*(t_{j-1}, t_j) & \prod_{j=4}^{k-1} a^*(t_{j-1}, t_j) & \cdots \\ \prod_{j=1}^{k} a^*(t_{j-1}, t_j) & \prod_{j=2}^{k} a^*(t_{j-1}, t_j) & \prod_{j=3}^{k} a^*(t_{j-1}, t_j) & \prod_{j=4}^{k} a^*(t_{j-1}, t_j) & \cdots \end{pmatrix}$$

$$
\begin{pmatrix}
\cdots & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 \\
\ddots & \vdots & \vdots & \vdots \\
\cdots & \prod_{j=k-1}^{k-1} a^*(t_{j-1}, t_j) & I_d & 0 \\
\cdots & \prod_{j=k-1}^{k} a^*(t_{j-1}, t_j) & \prod_{j=k}^{k} a^*(t_{j-1}, t_j) & I_d
\end{pmatrix}.
$$

And we find

$$
A \cdot \begin{pmatrix} X(t_0) \\ X(t_1) - a\,(t_0, t_1) * X(t_0) \\ X(t_2) - a\,(t_1, t_2) * X(t_1) \\ \vdots \\ X(t_k) - a\,(t_{k-1}, t_k) * X(t_{k-1}) \end{pmatrix} = \begin{pmatrix} X(t_0) \\ X(t_1) \\ X(t_2) \\ \vdots \\ X(t_k) \end{pmatrix}.
$$

We conclude that all finite-dimensional distributions of X are normal, and moreover, X satisfies all the mentioned assumptions in this theorem indeed. (See also p. 158, and p. 159 in Billingsley [1968]).
Let us compute $E\left(X^l(s)X^l(t)\right)$, for $0 \leqslant s \leqslant t \leqslant 1$, and $l = 1, \cdots, d$.
For this, we deal with the random array $\left(X^l(s), X^l(t)\right)^t$.

$$
\begin{pmatrix} X^l(s) \\ X^l(t) - a_l(s,t)X^l(s) \end{pmatrix} \sim \mathcal{N}\left(0, \begin{pmatrix} b_l^2(0,s) & 0 \\ 0 & b_l^2(s,t) \end{pmatrix}\right),
$$

where we can write

$$
b_l^2(\alpha, \beta) := \int_\alpha^\beta \sigma_l^2(r) \exp\left(2 \int_r^\beta \rho_l(\tau) d\tau\right) dr, \;\; l = 1, \cdots, d, \;\text{ and } \alpha, \beta \in [0,1].
$$

This will lead to

$$
\begin{pmatrix} X^l(s) \\ X^l(t) \end{pmatrix} \sim \mathcal{N}\left(0, B \begin{pmatrix} b_l^2(0,s) & 0 \\ 0 & b_l^2(s,t) \end{pmatrix} B^t\right),
$$

where

$$
B := \begin{pmatrix} 1 & 0 \\ a_l(s,t) & 1 \end{pmatrix},
$$

and

$$
\begin{pmatrix} 1 & 0 \\ a_l(s,t) & 1 \end{pmatrix} \begin{pmatrix} b_l^2(0,s) & 0 \\ 0 & b_l^2(s,t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_l(s,t) & 1 \end{pmatrix}^t =
$$

$$= \begin{pmatrix} b_l^2(0,s) & a_l(s,t)b_l^2(0,s) \\ a_l(s,t)b_l^2(0,s) & a_l^2(s,t)b_l^2(0,s) + b_l^2(s,t) \end{pmatrix}.$$

But from the forms of $a_l(\alpha, \beta)$, $b_l^2(\alpha, \beta)$, we find

$$a_l^2(s,t)b_l^2(0,s) + b_l^2(s,t) = b_l^2(0,t),$$

$$a_l(s,t)b_l^2(0,s) = \int_0^s \sigma_l^2(r) \exp\left(\int_r^s \rho_l(\tau)d\tau + \int_r^t \rho_l(\tau)d\tau \right) dr.$$

Therefore, the proof is complete. □

Let us here explain how to compute ρ_l, and σ_l^2, for $l = 1, 2, \cdots, d$. For this we put:

$$f_l(s,t) := E\left(X^l(s)X^l(t)\right), \text{ for all } 0 \leqslant s \leqslant t \leqslant 1, \text{ and } l = 1, 2, \cdots, d.$$

And consequently, if there exist ρ, and σ^2 suitable for X, then they satisfy the equalities:

$$\rho_l(t) = \frac{1}{f_l(s,t)} \frac{\partial f_l}{\partial t}(s,t),$$

for all $0 \leqslant s \leqslant t \leqslant 1$, and $l = 1, 2, \cdots, d$, and also

$$\sigma_l^2(s) = \left(\frac{\partial f_l}{\partial s}(s,t) - \rho_l(s) f_l(s,t) \right) / \exp\left(\int_s^t \rho_l(\tau)d\tau \right),$$

for all $0 \leqslant s \leqslant t \leqslant 1$, and $l = 1, 2, \cdots, d$.

Theorem 2.2.6. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random elements defined on $(\Omega, \mathcal{A}, P)$ and taking values in $(D[0,1])^d$. Suppose that the sequences $\left(\left(X_n^l(t)\right)^2\right)_{n\in\mathbb{N}}$, for $l = 1, 2, \cdots, d$, $t \in [0,1]$ are uniformly integrable, and $X_n(0) \xrightarrow{P} 0$. Suppose further that the functions ρ, σ^2 are defined and continuous on $[0,1)$ and taking values in $\mathbb{R}^d$.

$$\rho := (\rho_1, \cdots, \rho_d)^t, \ \sigma^2 := \left(\sigma_1^2, \cdots, \sigma_d^2\right)^t,$$

and for each l, the function σ_l^2 is nonnegative. Let us put for $t \in [0,1)$

$$g_l(t) := \exp\left(\int_0^t \rho_l(\tau)d\tau \right), \ l = 1, 2, \cdots, d,$$

$$G_l(t) := \int\limits_0^t \sigma_l^2(r) g_l^{-2}(r)dr,\ l = 1, 2, \cdots, d.$$

Suppose that the limits $\lim\limits_{t \longrightarrow 1} g_l(t)G_l(t)$ exist and are finite.
Suppose also that, for each positive ε, and η, there exists a positive δ such that

$$P\left\{w(X_n, \delta) \geqslant \varepsilon\right\} \leqslant \eta,\ \forall n \geqslant n_0$$

is valid for some $n_0 \in \mathbb{N}$ large enough. Where

$$w(x, \delta) := \sup_{|s-t|<\delta} |x(s) - x(t)|,\ x = (x^1, \cdots, x^d)^t \in (D[0,1])^d,$$

$0 < \delta \leqslant 1$, $s, t \in [0, 1]$, and $|x(s) - x(t)| := \max\limits_{1 \leqslant l \leqslant d} |x^l(s) - x^l(t)|$.
Suppose finally that $(X_n)_{n\in\mathbb{N}}$ satisfies the conditions $[1^o]$ (or $[1^o a]$), $[2^o]$ (or $[2^o a]$), and $[3^o]$ (or $[3^o a]$), where
$[1^o]$ If $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k < 1$, then

[I] $$\lim_{h\downarrow 0} \limsup_{n\longrightarrow\infty} \tfrac{1}{h} E\left\{\left| E\left\{X_n^l(t_k+h) - X_n^l(t_k) \middle| X_n(t_1), X_n(t_2), \cdots, X_n(t_k)\right\} - h\rho_l(t_k)X_n^l(t_k)\right|\right\} = 0, \text{ for } l = 1, \cdots, d,$$

[II] $$\lim_{h\downarrow 0} \limsup_{n\longrightarrow\infty} \tfrac{1}{h} E\left\{\left| E\left\{\left(X_n^l(t_k+h) - X_n^l(t_k)\right)^2 \middle| X_n(t_1), X_n(t_2), \cdots, X_n(t_k)\right\} - h\sigma_l^2(t_k)\right|\right\} = 0, \text{ for } l = 1, \cdots, d,$$

and

[III] $$\lim_{h\downarrow 0} \limsup_{n\longrightarrow\infty} \tfrac{1}{h} E\left\{\left| E\left\{ \left(X_n^{l_1}(t_k+h) - X_n^{l_1}(t_k)\right)\left(X_n^{l_2}(t_k+h) - X_n^{l_2}(t_k)\right) \middle| X_n(t_1), X_n(t_2), \cdots, X_n(t_k)\right\}\right|\right\} = 0, \text{ for } l_1 \neq l_2.$$

$[1^o a]$ If $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k < 1$, then
for all $\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_k \in \mathbb{R}^d$

[I] $$\lim_{h\downarrow 0} \limsup_{n\longrightarrow\infty} \tfrac{1}{h} E\left\{\left(\exp i \sum_{j=1}^k \tilde{u}_j^t X_n(t_j)\right)\left(X_n^l(t_k+h) - X_n^l(t_k) - h\rho_l(t_k)X_n^l(t_k)\right)\right\} = 0, \text{ for } l = 1, \cdots, d,$$

[II] $$\lim_{h\downarrow 0} \limsup_{n\longrightarrow\infty} \tfrac{1}{h} E\left\{\left(\exp i \sum_{j=1}^k \tilde{u}_j^t X_n(t_j)\right)\left(\left(X_n^l(t_k+h) - X_n^l(t_k)\right)^2 - h\sigma_l^2(t_k)\right)\right\} = 0, \text{ for } l = 1, \cdots, d,$$

and

[III] $\lim_{h\downarrow 0} \limsup_{n\longrightarrow\infty} \frac{1}{h} E\Bigg\{\left(\exp i \sum_{j=1}^{k} \tilde{u}_j^t X_n(t_j)\right) \left(X_n^{l_1}(t_k+h) - X_n^{l_1}(t_k)\right)$

$$\left(X_n^{l_2}(t_k+h) - X_n^{l_2}(t_k)\right)\Bigg\} = 0, \text{ for } l_1 \neq l_2.$$

[2^o] $\sup_{t\in[0,1]} \limsup_{n\longrightarrow\infty} E\left(\left(X_n^l(t)\right)^2\right) < \infty$, for $l = 1, 2, \cdots, d$.

[$2^o a$] $\lim_{\alpha\longrightarrow\infty} \sup_{t\in[0,1]} \limsup_{n\longrightarrow\infty} \int_{\left\{\left|X_n^l(t)\right|>\alpha\right\}} \left|X_n^l(t)\right| dP = 0$, for $l = 1, 2, \cdots, d$.

[3^o] There is a constant K such that for each $l = 1, \cdots, d$

$$\limsup_{n\longrightarrow\infty} E\left\{\left(X_n^l(t) - X_n^l(t_1)\right)^2 \left(X_n^l(t_2) - X_n^l(t)\right)^2\right\} < K\left(t_2 - t_1\right)^2,$$

for all t_1, t, t_2 satisfying $0 \leqslant t_1 \leqslant t \leqslant t_2 \leqslant 1$.

[$3^o a$] $\lim_{\alpha\longrightarrow\infty} \limsup_{h\downarrow 0} \limsup_{n\longrightarrow\infty} \int_{\left\{\left(X_n^l(t+h)-X_n^l(t)\right)^2 \geqslant \alpha h\right\}} \left(X_n^l(t+h) - X_n^l(t)\right)^2 dP = 0,$

for each $l = 1, \cdots, d$, and for all $0 \leqslant t < 1$.
Then $X_n \xrightarrow{\mathcal{D}} X$, where X is the continuous Gaussian function specified by

$$E(X(t)) = 0, \ \forall t \in [0, 1],$$

and for $0 \leqslant s \leqslant t \leqslant 1$

$$E\left(X^{l_1}(s)X^{l_2}(t)\right) = \begin{cases} \int_0^s \sigma_l^2(r) \exp\left(\int_r^s \rho_l(\tau)d\tau + \int_r^t \rho_l(\tau)d\tau\right) dr & , \text{for } l_1 = l_2 \\ 0 & , \text{for } l_1 \neq l_2 \end{cases}.$$

More specifically,

$$(X(t))_{0\leqslant t\leqslant 1} \sim \begin{pmatrix} g_1(t)\left(1 + G_1(t)\right) W^{o1}_{\left(\frac{G_1(t)}{1+G_1(t)}\right)} \\ \vdots \\ g_d(t)\left(1 + G_d(t)\right) W^{od}_{\left(\frac{G_d(t)}{1+G_d(t)}\right)} \end{pmatrix}_{0\leqslant t\leqslant 1},$$

where here, the random variables W^{o1}, W^{o2}, $\cdots$, W^{od}, are independent one-dimensional Brownian bridges, and also we put

$$\frac{G_l(1)}{1 + G_l(1)} := \lim_{t\longrightarrow 1} \frac{G_l(t)}{1 + G_l(t)}, \ l = 1, 2, \cdots, d,$$

and

$$g_l(1)\left(1 + G_l(1)\right) := \lim_{t\longrightarrow 1} g_l(t)\left(1 + G_l(t)\right), \ l = 1, 2, \cdots, d.$$

Proof: Similarly to the proof of theorem 19.4 in Billingsley [1968], we can write here the following arguments:
$(1^o) \Longrightarrow (1^o a)$, and $(2^o) \Longrightarrow (2^o a)$, but the condition (3^o) does not imply the condition $(3^o a)$ in general, since from the assumption we have

$$P\left\{X_n \notin (C[0,1])^d\right\} > 0, \text{ probably for infinitly many } n.$$

(See also p. 156, p. 160, and p. 164 in Billingsley [1968]).
By theorem 15.5 in the same mentioned reference above and from the assumptions, we conclude that each subsequence of $(X_n)_{n\in\mathbb{N}}$ contains a further sub-subsequence, such that, it is convergent in distribution to some random element X of $(D[0,1])^d$, with $P\left\{X \in (C[0,1])^d\right\} = 1$, and $X(0) = 0$ almost sure under P. Also similarly, we find

$$(1^o a) \Longrightarrow (1a),\ (2^o a) \Longrightarrow (2a),\ (3^o) \Longrightarrow (3) \Longrightarrow (3a),\ (3^o a) \Longrightarrow (3a).$$

In other words, the random element X satisfies the conditions $(1a)$, $(2a)$, and $(3a)$.
And this completes the proof.
As a consequence of the remark and the computations in p. 165 in Billingsley [1968], we see that the following conditions are also sufficient to come to the same conclusion of the previous theorem (1^o) (or $(1^o a)$), (3^o), $P\{X_n(0) = 0\} = 1,\ \forall n \in \mathbb{N}$, and

$$\lim_{n\longrightarrow\infty} \max_{0<t\leqslant 1} \left\| X_n(t) - \lim_{h\downarrow 0} X_n(t-h) \right\| = 0\ [P].$$

Where $\|\cdot\|$ is the usual norm defined on $\mathbb{R}^d$. □

2.3. Applications:

In this section we present some examples to show the reader how one can apply the limit theorems, which are discussed in this chapter. And the purpose of thses examples is mainly to establish the needed base for building the proofs of another limit theorems related with the conditional distributions of specific linear permutation statistics similar to those in chapter three. And the following theorem is intended to be used later when we shall deal with permutation statistics for the hypothesis of randomness H_0.

Theorem 2.3.1. For each $n \in \mathbb{N}$, let $x_{n1}, x_{n2}, \cdots, x_{nk_n}$ be a sequence of d-dimensional arrays of real numbers which are not distinct necessarily, let further $(\xi_{n1}, \xi_{n2}, \cdots, \xi_{nk_n})$ be an array of random arrays, and taking values in the set of all the permutations of $(x_{n1}, x_{n2}, \cdots, x_{nk_n})$ which is $\{\pi(x_{n1}, x_{n2}, \cdots, x_{nk_n}) : \pi \in \Pi_{k_n}\}$, where Π_{k_n} is the symmetric group of the permutation functions of order k_n, and we suppose further that each one of the $k_n!$ permutations having probability $1/k_n!$. And let us define X_n by $X_n(t) := \sum_{i=1}^{[k_n t]} \xi_{ni}$, $X_n(t) := 0$ whenever $0 \leqslant t < 1/k_n$, where $[\cdot]$ stands for the integral value function. Then we assert that:
If $\sum_{i=1}^{k_n} x_{ni} x_{ni}^t = I_d$, $\sum_{i=1}^{k_n} x_{ni} = 0$, and $\max_{1 \leqslant i \leqslant k_n} \parallel x_{ni} \parallel \underset{n \longrightarrow \infty}{\longrightarrow} 0$, then we have also $X_n \overset{\mathcal{D}}{\longrightarrow} W^o$, where W^o here is a d-dimensional Brownian bridge.

Proof: In fact, it is enough to verify the conditions (1^o) and (3^o) in theorem 2.2.6, for $\rho_l(t) = -1/(1-t)$, $\sigma_l^2(t) = 1$, $l = 1, \cdots, d$, $t \in [0, 1)$, that characterize the d-Brownian Bridge W^o, and to verify the remark following the proof of that theorem. By doing that, we achieve the assertion of this theorem. We shall verify the condition (1^o) , then the mentioned remark, and next we turn to the condition (3^o).
Let us now begin with the folloing preliminary computations:
Easily, one can see that $E(\xi_{n1}) = E(\xi_{n2}) = \cdots = E(\xi_{nk_n}) = 0$. and

$$E\left(\xi_{n1}\left(\xi_{n1}\right)^t\right) = E\left(\xi_{n2}\left(\xi_{n2}\right)^t\right) = \cdots = E\left(\xi_{nk_n}\left(\xi_{nk_n}\right)^t\right) = \frac{1}{k_n} I_d,$$

$$E\left(\xi_{ni}\left(\xi_{nj}\right)^t\right) = E\left(\xi_{n1}\left(\xi_{n2}\right)^t\right) = \frac{-1}{k_n(k_n-1)} I_d,$$

whenever $1 \leqslant i \neq j \leqslant k_n$. Also,

$$E\left(\sum_{i=1}^{[k_nt]} \xi_{ni}\right) = [k_nt]E\left(\xi_{n1}\right) = 0,$$

and

$$E\left\{\left(\sum_{i=1}^{[k_nt]} \xi_{ni}\right)\left(\sum_{i=1}^{[k_nt]} \xi_{ni}\right)^t\right\} = [k_nt]E\left(\xi_{n1}\left(\xi_{n1}\right)^t\right) +$$
$$+[k_nt]([k_nt]-1)E\left(\xi_{n1}\left(\xi_{n2}\right)^t\right)$$
$$= \left(\frac{[k_nt]}{k_n} - \frac{[k_nt]([k_nt]-1)}{k_n(k_n-1)}\right) I_d \longrightarrow t(1-t)I_d \text{ for } n \longrightarrow \infty.$$

Also, we need (as we shall see in the proof) to consider the limit

$$\lim_{n\longrightarrow\infty} \sum_{i=1}^{k_n} \left(x_{ni}^l\right)^4 \leqslant \lim_{n\longrightarrow\infty} \sum_{i=1}^{k_n} \left(x_{ni}^l\right)^2 \left(\max_{1\leqslant i\leqslant k_n} \left|x_{ni}^l\right|\right)^2 = 0, \text{ for } l = 1, \cdots, d.$$

So far, we have achieved

$E\left(X_n(t)\right) = 0$, $E\left(X_n(t)\left(X_n(t)\right)^t\right) = \frac{[k_nt](k_n-[k_nt])}{k_n(k_n-1)} \longrightarrow t(1-t)I_d$ for $n \longrightarrow \infty$.

By noticing these computations, one can guess the form of $\rho(t)$ and $\sigma^2(t)$ which could serve in proving the assertion of this theorem. Now, we begin verifying the mentioned condition (1^o) which states:

[1^o] If $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k < 1$, then

[I] $$\lim_{h\downarrow 0}\limsup_{n\longrightarrow\infty} \frac{1}{h} E\left\{\left|E\left\{X_n^l(t_k+h) - X_n^l(t_k)\middle| X_n(t_1), X_n(t_2), \cdots, X_n(t_k)\right\} - \right.\right.$$
$$\left.\left. -h\rho_l(t_k)X_n^l(t_k)\right|\right\} = 0, \text{ for } l = 1, \cdots, d,$$

[II] $$\lim_{h\downarrow 0}\limsup_{n\longrightarrow\infty} \frac{1}{h} E\left\{\left|E\left\{\left(X_n^l(t_k+h) - X_n^l(t_k)\right)^2 \middle| X_n(t_1), X_n(t_2), \cdots, X_n(t_k)\right\}\right.\right.$$
$$\left.\left. -h\sigma_l^2(t_k)\right|\right\} = 0, \text{ for } l = 1, \cdots, d,$$

and

[III] $$\lim_{h\downarrow 0}\limsup_{n\longrightarrow\infty} \frac{1}{h} E\left\{\left|E\left\{\left(X_n^{l_1}(t_k+h) - X_n^{l_1}(t_k)\right)\left(X_n^{l_2}(t_k+h) - X_n^{l_2}(t_k)\right) \right|\right.\right.$$
$$\left.\left.\left| X_n(t_1), X_n(t_2), \cdots, X_n(t_k)\right\}\right|\right\} = 0, \text{ for } l_1 \neq l_2.$$

Let us verify (I): Since $X_n(0) = 0$ is valid and $t_k \leqslant t < 1$ must be hold, the case $t = 0$ is trivial. Fix $0 < t < 1$, for reasons of convenience, we put $m_1 := [k_nt]$, and $m_2 := [k_n(t+h)] - [k_nt]$, where $0 < h \leqslant 1-t$. Consequently, for large n we have $1 \leqslant m_1 < m_1 + m_2 \leqslant k_n$.

Suppose that we know the values of $\xi_{n1}, \cdots, \xi_{nm_1}$, then $\xi_{n,m_1+1}, \cdots, \xi_{n,m_1+m_2}$ are conditionally distributed as a sample of points of size m_2 taken from the population $x_{n1}, x_{n2}, \cdots, x_{nk_n}$ after removing the points corresponding to the

sample $\xi_{n1},\cdots,\xi_{nm_1}$. And we have similarly to the beginning of the proof:
$E\left\{\sum_{i=m_1+1}^{m_1+m_2}\xi_{ni}\big|\xi_{n1},\cdots,\xi_{nm_1}\right\}=-\frac{m_2}{k_n-m_1}\sum_{i=1}^{m_1}\xi_{ni}$, which implies
$E\left\{X_n(t+h)-X_n(t)\big|\xi_{n1},\cdots,\xi_{n[k_nt]}\right\}=-\frac{[k_n(t+h)]-[k_nt]}{k_n-[k_nt]}\sum_{i=1}^{[k_nt]}\xi_{ni}$.
And we can rewrite it as
$E\left\{X_n(t+h)-X_n(t)\big|\xi_{n1},\cdots,\xi_{n[k_nt]}\right\}=-\frac{[k_n(t+h)]-[k_nt]}{k_n-[k_nt]}X_n(t)$.
Now, for $t=t_k$, and since

$$\sigma(X_n(t_1),\cdots,X_n(t_k))\subseteq\sigma(\xi_{n1},\cdots,\xi_{n[k_nt_k]}),$$

we have,
$E\left\{X_n\left(t_k+h\right)-X_n\left(t_k\right)\big|X_n\left(t_1\right),\cdots,X_n\left(t_k\right)\right\}=-\frac{[k_n(t_k+h)]-[k_nt_k]}{k_n-[k_nt_k]}X_n(t_k)$.
And for $l=1,\cdots,d$ we find
$\frac{1}{h}E\left\{\left|E\left\{X_n^l(t_k+h)-X_n^l(t_k)\big|X_n(t_1),\cdots,X_n(t_k)\right\}-h\cdot\frac{-1}{1-t_k}\cdot X_n^l(t_k)\right|\right\}=$

$$=\frac{1}{h}\left|\underbrace{\frac{[k_n(t_k+h)]-[k_nt_k]}{k_n-[k_nt_k]}}_{\underset{n\longrightarrow\infty}{\longrightarrow}\frac{h}{1-t_k}}-\frac{h}{1-t_k}\right|\cdot E\left\{\left|X_n^l(t_k)\right|\right\}.$$

But from the computations at the beginning of this proof, we conclude that

$$E\left\{\left|X_n^l(t_k)\right|\right\}\leqslant\left(E\left\{\left(X_n^l(t_k)\right)^2\right\}\right)^{\frac{1}{2}}<\text{Constant},$$

$\forall n\in\mathbb{N}$, and for $l=1,\cdots,d$.
Hence, (I) is verified clearly.
Also, for $l=1,\cdots,d$, we have

$$E\left\{\left(\sum_{i=m_1+1}^{m_1+m_2}\xi_{ni}^l\right)^2\big|\xi_{n1},\cdots,\xi_{nm_1}\right\}=$$
$$=m_2E\left\{\left(\xi_{n,m_1+1}^l\right)^2\big|\xi_{n1},\cdots,\xi_{nm_1}\right\}+$$
$$+m_2(m_2-1)E\left\{\xi_{n,m_1+1}^l\xi_{n,m_1+2}^l\big|\xi_{n1},\cdots,\xi_{nm_1}\right\}$$

$$=\frac{m_2(k_n-m_1-m_2)}{(k_n-m_1)(k_n-m_1-1)}\left(1-\sum_{i=1}^{m_1}(\xi_{ni}^l)^2\right)+\frac{m_2(m_2-1)}{(k_n-m_1)(k_n-m_1-1)}\left(\sum_{i=1}^{m_1}\xi_{ni}^l\right)^2.$$

Therefore, for $t=t_k$ we find

$$E\left\{\left(X_n^l(t_k+h)-X_n^l(t_k)\right)^2\big|\xi_{n1},\cdots,\xi_{n[k_nt_k]}\right\}=$$
$$=B_n(h)\left(1-\sum_{i=1}^{[k_nt_k]}(\xi_{ni}^l)^2\right)+C_n(h)\left(\sum_{i=1}^{[k_nt_k]}\xi_{ni}^l\right)^2,$$

where

$$B_n(h) := \frac{([k_n(t_k+h)]-[k_nt_k])\,(k_n-[k_n(t_k+h)])}{(k_n-[k_nt_k])\,(k_n-[k_nt_k]-1)},$$

and

$$C_n(h) := \frac{([k_n(t_k+h)]-[k_nt_k])\,(([k_n(t_k+h)]-[k_nt_k])-1)}{(k_n-[k_nt_k])\,(k_n-[k_nt_k]-1)}.$$

Since

$$\sigma(X_n(t_1),\cdots,X_n(t_k)) \subseteq \sigma(\xi_{n1},\cdots,\xi_{n[k_nt_k]}),$$

and from chapter one in this dissertation we have

$$\mathcal{C}_1 \subseteq \mathcal{C}_2 \Longrightarrow E\left\{|E(f|\mathcal{C}_1)|\right\} \leqslant E\left\{|E(f|\mathcal{C}_2)|\right\}.$$

So, we can write here

$$\tfrac{1}{h}E\left\{\left|E\left\{\left(X_n^l(t_k+h)-X_n^l(t_k)\right)^2\left|X_n(t_1),\cdots,X_n(t_k)\right.\right\}-h\cdot 1\right|\right\} \leqslant$$
$$\leqslant \tfrac{1}{h}E\left\{\left|E\left\{\left(X_n^l(t_k+h)-X_n^l(t_k)\right)^2\left|\xi_{n1},\cdots,\xi_{n[k_nt_k]}\right.\right\}-h\cdot 1\right|\right\} \leqslant$$
$$\leqslant \tfrac{|B_n(h)|}{h}E\left\{\left|\sum_{i=1}^{[k_nt_k]}(\xi_{ni}^l)^2-\tfrac{[k_nt_k]}{k_n}\right|\right\}+\left|\tfrac{B_n(h)}{h}\left(1-\tfrac{[k_nt_k]}{k_n}\right)-1\right|+$$
$$+\tfrac{|C_n(h)|}{h}E\left\{\left(X_n^l(t_k)\right)^2\right\}.$$

But also we see that

$$B_n(h) \underset{n\longrightarrow\infty}{\longrightarrow} \frac{h(1-t_k-h)}{(1-t_k)^2},$$

$$C_n(h) \underset{n\longrightarrow\infty}{\longrightarrow} \frac{h^2}{(1-t_k)^2},$$

$$\left|\frac{B_n(h)}{h}\left(1-\frac{[k_nt_k]}{k_n}\right)-1\right| \underset{n\longrightarrow\infty}{\longrightarrow} \frac{h}{(1-t_k)},$$

and

$$E\left\{\left(X_n^l(t_k)\right)^2\right\} < \text{constant},\ \forall n\in\mathbb{N}.$$

So, to verify (II) it remains to prove the validity of the limit

$$E\left\{\left|\sum_{i=1}^{[k_nt_k]}(\xi_{ni}^l)^2-\frac{[k_nt_k]}{k_n}\right|\right\} \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

For this, and from the beginning of the proof we can write that

$$E\left(\sum_{i=1}^{[k_nt_k]}\left(\xi_{ni}^l\right)^2\right) = \frac{[k_nt_k]}{k_n} \underset{n\longrightarrow\infty}{\longrightarrow} t_k,$$

and let us here compute the second moment of $\sum\limits_{i=1}^{[k_n t_k]} \left(\xi_{ni}^l\right)^2$.

$E\left(\sum\limits_{i=1}^{[k_n t_k]} \left(\xi_{ni}^l\right)^2\right)^2 = \frac{[k_n t_k](k_n-[k_n t_k])}{k_n(k_n-1)} \underbrace{\sum_{i=1}^{k_n} (x_{ni}^l)^4}_{\underset{n\longrightarrow\infty}{\longrightarrow} 0} + \frac{[k_n t_k]([k_n t_k]-1)}{k_n(k_n-1)} \underset{n\longrightarrow\infty}{\longrightarrow} t_k^2.$

But this implies $Var\left(\sum\limits_{i=1}^{[k_n t_k]} \left(\xi_{ni}^l\right)^2\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0.$

And consequently, the limit

$$E\left\{\left|\sum_{i=1}^{[k_n t_k]} (\xi_{ni}^l)^2 - \frac{[k_n t_k]}{k_n}\right|\right\} \underset{n\longrightarrow\infty}{\longrightarrow} 0$$

is valid indeed.

Therefore, also the condition (II) is verified.

Now, let us verify the condition (III).

For $1 \leqslant l_1 \neq l_2 \leqslant d$, we have

$E\left\{\left(\sum\limits_{i=m_1+1}^{m_1+m_2} \xi_{ni}^{l_1}\right)\left(\sum\limits_{i=m_1+1}^{m_1+m_2} \xi_{ni}^{l_2}\right)\big|\xi_{n1},\cdots,\xi_{nm_1}\right\} =$

$= m_2 E\left\{\xi_{n,m_1+1}^{l_1}\xi_{n,m_1+1}^{l_2}\big|\xi_{n1},\cdots,\xi_{nm_1}\right\} +$

$+m_2(m_2-1)E\left\{\xi_{n,m_1+1}^{l_1}\xi_{n,m_1+2}^{l_2}\big|\xi_{n1},\cdots,\xi_{nm_1}\right\}$

$= -\frac{m_2(k_n-m_1-m_2)}{(k_n-m_1)(k_n-m_1-1)}\left(\sum\limits_{i=1}^{m_1} \xi_{ni}^{l_1}\xi_{ni}^{l_2}\right) + \frac{m_2(m_2-1)}{(k_n-m_1)(k_n-m_1-1)}\left(\sum\limits_{i=1}^{m_1} \xi_{ni}^{l_1}\right)\left(\sum\limits_{i=1}^{m_1} \xi_{ni}^{l_2}\right).$

Therefore, for $t = t_k$ we find

$E\left\{\left(X_n^{l_1}(t_k+h) - X_n^{l_1}(t_k)\right)\left(X_n^{l_2}(t_k+h) - X_n^{l_2}(t_k)\right)\big|\xi_{n1},\cdots,\xi_{n[k_n t_k]}\right\} =$

$= -B_n(h)\left(\sum\limits_{i=1}^{[k_n t_k]} \xi_{ni}^{l_1}\xi_{ni}^{l_2}\right) + C_n(h)\left(\sum\limits_{i=1}^{[k_n t_k]} \xi_{ni}^{l_1}\right)\left(\sum\limits_{i=1}^{[k_n t_k]} \xi_{ni}^{l_2}\right),$

$\frac{1}{h}E\left\{\left|E\left\{\left(X_n^{l_1}(t_k+h) - X_n^{l_1}(t_k)\right)\left(X_n^{l_2}(t_k+h) - X_n^{l_2}(t_k)\right)\big|\xi_{n1},\cdots,\xi_{n[k_n t_k]}\right\}\right|\right\}$

$\leqslant \frac{|B_n(h)|}{h}E\left(\left|\sum\limits_{i=1}^{[k_n t_k]} \xi_{ni}^{l_1}\xi_{ni}^{l_2}\right|\right) + \frac{|C_n(h)|}{h}E\left(\left|\sum\limits_{i=1}^{[k_n t_k]} \xi_{ni}^{l_1}\right|\left|\sum\limits_{i=1}^{[k_n t_k]} \xi_{ni}^{l_2}\right|\right)$

So, to verify (III) it remains to prove the validity of the limit

$$E\left(\left|\sum_{i=1}^{[k_n t_k]} \xi_{ni}^{l_1}\xi_{ni}^{l_2}\right|\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

For this, let us compute the moment of $\sum\limits_{i=1}^{[k_n t_k]} \xi_{ni}^{l_1}\xi_{ni}^{l_2}$

$$E\left(\sum_{i=1}^{[k_nt_k]}\xi_{ni}^{l_1}\xi_{ni}^{l_2}\right)^2 = \tfrac{[k_nt_k](k_n-[k_nt_k])}{k_n(k_n-1)}\underbrace{\sum_{i=1}^{k_n}(x_{ni}^{l_1})^2(x_{ni}^{l_2})^2}_{\underset{n\longrightarrow\infty}{\longrightarrow}0} + \tfrac{[k_nt_k]([k_nt_k]-1)}{k_n(k_n-1)}\underbrace{\left(\sum_{i=1}^{k_n}x_{ni}^{l_1}x_{ni}^{l_2}\right)^2}_{=0}.$$

And hence, we have verified the condition (1^o).
Now, we turn to the second part of the proof. So, let us verifiy the remark, mentioned at the beginning of the proof:
Since the maximum jump $\max_{1\leqslant i\leqslant k_n}\parallel x_{ni}\parallel$ in X_n tends to zero and $X_n(0)=0, \forall n\in\mathbb{N}$, the mentioned remark is verified obviously.
Now, we turn to condition(3^o) which states:
There is a constant K such that, for all $l=1,\cdots,d$, and for all $t_1,t,t_2\in[0,1]$ satisfying $t_1\leqslant t\leqslant t_2$, then
$\limsup_{n\longrightarrow\infty} E\left\{\left(X_n^l(t)-X_n^l(t_1)\right)^2\left(X_n^l(t_2)-X_n^l(t)\right)^2\right\}\leqslant K(t_2-t_1)^2.$
First, if $t_2-t_1\leqslant\frac{1}{k_n}$, then $E\left\{\left(X_n^l(t)-X_n^l(t_1)\right)^2\left(X_n^l(t_2)-X_n^l(t)\right)^2\right\}=0$, and the condition holds.
Now, if $t_2-t_1>\frac{1}{k_n}$, then we find easily that:

$$E\left\{\left(X_n^l(t)-X_n^l(t_1)\right)^2\left(X_n^l(t_2)-X_n^l(t)\right)^2\right\}=\sum\{\xi_{ni}^l\xi_{nj}^l\xi_{nk}^l\xi_{nl}^l\},$$

where $[k_nt_1]<i,j\leqslant[k_nt]$, and $[k_nt]<k,l\leqslant[k_nt_2]$. Let us put here $m_1:=[k_nt]-[k_nt_1]$ and $m_2:=[k_nt_2]-[k_nt]$. We can write for n large enough

$$Q_n^l := \sum\{\xi_{ni}^l\xi_{nj}^l\xi_{nk}^l\xi_{nl}^l\} =$$
$$= m_1m_2E\{\left(\xi_{n1}^l\right)^2\left(\xi_{n2}^l\right)^2\} + m_1m_2(m_1+m_2-2)E\{\left(\xi_{n1}^l\right)^2\xi_{n2}^l\xi_{n3}^l\} + m_1(m_1-1)m_2(m_2-1)E\{\xi_{n1}^l\xi_{n2}^l\xi_{n3}^l\xi_{n4}^l\}.$$

$$= m_1m_2\tfrac{1-\tau_n^l}{k_n(k_n-1)} + m_1m_2(m_1+m_2-2)\tfrac{2\tau_n^l-1}{k_n(k_n-1)(k_n-2)} + m_1(m_1-1)m_2(m_2-1)\tfrac{3(1-2\tau_n^l)}{k_n(k_n-1)(k_n-2)(k_n-3)},$$

where $\tau_n^l := \sum_{i=1}^{k_n}(x_{ni}^l)^4\leqslant 1$.
$\limsup_{n\longrightarrow\infty} Q_n^l\leqslant\frac{11}{16}\left(t_2-t_1\right)^2$, and this is since we have $m_1m_2\leqslant\frac{1}{4}\left(m_1+m_2\right)^2$, and $\frac{m_1+m_2}{k_n}\underset{n\longrightarrow\infty}{\longrightarrow}t_2-t_1$.
This will lead to the validity of the limit

$$\limsup_{n\longrightarrow\infty} E\left\{\left(X_n^l(t)-X_n^l(t_1)\right)^2\left(X_n^l(t_2)-X_n^l(t)\right)^2\right\}\leqslant\frac{11}{16}(t_2-t_1)^2.$$

Therefore, also the condition (3^o) is verified for $K \geqslant \frac{11}{16}$, and consequently the proof is complete. $\square$

Remark 2.3.2. Under the same hypotheses of theorem 2.3.1, but here the sequences $\left(\sum_{i=1}^{k_n}(x_{ni}-\bar{x}_n)(x_{ni}-\bar{x}_n)^t\right)_{n\in\mathbb{N}}$, and $\left(\sum_{i=1}^{k_n} x_{ni}\right)_{n\in\mathbb{N}}$ are bounded, where $\bar{x}_n = \frac{1}{k_n}\sum_{i=1}^{k_n} x_{ni}$, also we assume that the matrices $s^2\left(\underline{x}_{k_n}\right)$ are positive definite. Where $s^2\left(\underline{x}_{k_n}\right) := \sum_{i=1}^{k_n}(x_{ni}-\bar{x}_n)(x_{ni}-\bar{x}_n)^t$. And we assume further that the triangular array $\underline{x}_{k_n}$ is infinitesimal, where $\underline{x}_{k_n} := (x_{n1},\cdots,x_{nk_n})$, then we have

$$(X_n(t))_{0\leqslant t\leqslant 1} \overset{w}{\sim} \left(\sqrt{s^2\left(\underline{x}_{k_n}\right)}W_t^o + t\sum_{i=1}^{k_n} x_{ni}\right)_{0\leqslant t\leqslant 1} \quad (\mathcal{C}_n).$$

Where the assertion means by definition

$$E(\varphi(X_n)) - E\left(\varphi\left(\sqrt{s^2\left(\underline{x}_{k_n}\right)}W^o + (\cdot)\sum_{i=1}^{k_n} x_{ni}\right)\right) \longrightarrow 0,$$

or more explicitly

$$\frac{1}{k_n!}\sum_{\pi}\varphi\left(\sum_{i=1}^{[k_n(\cdot)]} x_{n\pi(i)}\right) - \int\limits_{(C[0,1])^d}\varphi\left(\sqrt{s^2\left(\underline{x}_{k_n}\right)}W^o + (\cdot)\sum_{i=1}^{k_n} x_{ni}\right)d\mu \longrightarrow 0.$$

For all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where μ is the Wiener measure, defined $(C[0,1])^d$, and where π here ranging over all bijective functions, defined on $\{1,2,\cdots,k_n\}$.

Proof: It is sufficient to prove it in the case, where there exists $\varepsilon > 0$, such that $\parallel s^2(\underline{x}_{k_n}) \parallel > \varepsilon$, is valid $\forall n \in \mathbb{N}$.

Under the same hypotheses and by using computations similarly to those in the proof of theorem 2.3.1, one can conclude that $Var\left(X_n(t) - t\sum_{i=1}^{k_n} x_{ni}\right)$ will tend to zero if and only if $s^2\left(\underline{x}_{k_n}\right)$ tends to zero, and the assertion then is valid. See also lemma 1.2.4.

Now, to reduce the case of this remark to the case of theorem 2.3.1, we put $y_{ni} := \left(s^2(\underline{x}_{k_n})\right)^{-\frac{1}{2}}(x_{ni}-\bar{x}_n)$, $i = 1,\cdots,k_n$, $n \in \mathbb{N}$, then we have

$\left(\left(s^2(\underline{x}_{k_n})\right)^{-\frac{1}{2}}\sum_{i=1}^{[k_n t]}(\xi_{ni}-\bar{x}_n)\right)_{0\leqslant t\leqslant 1}\xrightarrow{\mathcal{D}}W^o$. Since the sequences $\left(s^2(\underline{x}_{k_n})\right)_{n\in\mathbb{N}}$, and $\left(\sum_{i=1}^{k_n} x_{ni}\right)_{n\in\mathbb{N}}$ are bounded, the assertion is valid. □

The following theorem is intended to be used later when we shall deal with permutation statistics for the hypothesis of symmetry H_1.

Theorem 2.3.3. For each $n\in\mathbb{N}$, let $x_{n1}, x_{n2}, \cdots, x_{nk_n}$ be a sequence of d-dimensional arrays of real numbers which are not distinct necessarily, let further $(\xi_{n1}, \xi_{n2}, \cdots, \xi_{nk_n})$ be an array of random arrays, and taking values in the set of all the permutations of $(x_{n1}, x_{n2}, \cdots, x_{nk_n})$ which is $\{\pi(x_{n1}, x_{n2}, \cdots, x_{nk_n}) : \pi \in \Pi_{k_n}\}$, where Π_{k_n} is the symmetric group of the permutation functions of order k_n, and we suppose further that each one of the $k_n!$ permutations having probability $1/k_n!$. Also, let $\varepsilon_{n1}, \varepsilon_{n2}, \cdots, \varepsilon_{nk_n}$ be i.i.d. variables, and taking values in $\{+1, -1\}$ such that each of $+1$ and -1 having probability 1/2, and these variables are independent of ξ_{n1}, ξ_{n2}, $\cdots$, ξ_{nk_n} too. Finally, let us define the random arrays $\zeta_{n1}, \zeta_{n2}, \cdots, \zeta_{nk_n}$ by $\zeta_{ni} := \varepsilon_{ni}\xi_{ni}, i = 1, \cdots, k_n$. And let us define X_n by $X_n(t) := \sum_{i=1}^{[k_n t]}\zeta_{ni}$, $X_n(t) := 0$ whenever $0 \leqslant t < 1/k_n$, where $[\cdot]$ stands for the integral value function. Then we assert that:
If $\sum_{i=1}^{k_n} x_{ni}x_{ni}^t = I_d$ and $\max_{1\leqslant i\leqslant k_n} \parallel x_{ni} \parallel \underset{n\longrightarrow\infty}{\longrightarrow} 0$, then we have also $X_n \xrightarrow{\mathcal{D}} W$, where W here is a d-dimensional Brownian motion.

Proof: In fact, it is enough to verify the conditions (1^o) and (3^o) in theorem 2.2.6, for $\rho_l(t) = 0$, $\sigma_l^2(t) = 1$, $l = 1, \cdots, d$, $t \in [0, 1)$, that characterize the d-Brownian motion W, and to verify the remark following the proof of that theorem. By doing that, we achieve the assertion of this theorem. We shall verify the condition (1^o) , then the mentioned remark, and next we turn to the condition (3^o).
Let us now begin with the folloing preliminary computations:
Easily, one can see that $E(\zeta_{n1}) = E(\zeta_{n2}) = \cdots = E(\zeta_{nk_n}) = 0.$ and

$$E\left(\zeta_{n1}(\zeta_{n1})^t\right) = E\left(\zeta_{n2}(\zeta_{n2})^t\right) = \cdots = E\left(\zeta_{nk_n}(\zeta_{nk_n})^t\right) = \frac{1}{k_n}I_d,$$

$$E\left(\zeta_{ni}(\zeta_{nj})^t\right) = E\left(\zeta_{n1}(\zeta_{n2})^t\right) = 0,$$

whenever $1 \leqslant i \neq j \leqslant k_n$. Also,

$$E\left(\sum_{i=1}^{[k_n t]}\zeta_{ni}\right) = [k_n t]E(\zeta_{n1}) = 0,$$

and
$E\left\{\left(\sum_{i=1}^{[k_nt]}\zeta_{ni}\right)\left(\sum_{i=1}^{[k_nt]}\zeta_{ni}\right)^t\right\} = [k_nt]E\left(\zeta_{n1}\left(\zeta_{n1}\right)^t\right) +$
$+[k_nt]([k_nt]-1)E\left(\zeta_{n1}\left(\zeta_{n2}\right)^t\right)$
$= \left(\frac{[k_nt]}{k_n}+0\right)I_d \longrightarrow tI_d$ for $n \longrightarrow \infty$.
Also, we need (as we shall see in the proof) to consider the limit
$\lim_{n\longrightarrow\infty}\sum_{i=1}^{k_n}\left(x_{ni}^l\right)^4 \leqslant \lim_{n\longrightarrow\infty}\sum_{i=1}^{k_n}\left(x_{ni}^l\right)^2\left(\max_{1\leqslant i\leqslant k_n}\left|x_{ni}^l\right|\right)^2 = 0$, for $l = 1,\cdots,d$.
So far, we have achieved
$E\left(X_n(t)\right) = 0,\ E\left(X_n(t)\left(X_n(t)\right)^t\right) = \frac{[k_nt]}{k_n} \longrightarrow tI_d$ for $n \longrightarrow \infty$.
By noticing these computations, one can guess the form of $\rho(t)$ and $\sigma^2(t)$ which could serve in proving the assertion of this theorem. Now, we begin verifying the mentioned condition (1^o) which states:
[1^o] If $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k < 1$, then
[I] $\lim_{h\downarrow 0}\limsup_{n\longrightarrow\infty}\frac{1}{h}E\left\{\left|E\left\{X_n^l(t_k+h)-X_n^l(t_k)\middle|X_n(t_1),X_n(t_2),\cdots,X_n(t_k)\right\}-\right.\right.$
$\left.\left.-h\rho_l(t_k)X_n^l(t_k)\right|\right\} = 0$, for $l = 1,\cdots,d$,
[II] $\lim_{h\downarrow 0}\limsup_{n\longrightarrow\infty}\frac{1}{h}E\left\{\left|E\left\{\left(X_n^l(t_k+h)-X_n^l(t_k)\right)^2\middle|X_n(t_1),X_n(t_2),\cdots,X_n(t_k)\right\}\right.\right.$
$\left.\left.-h\sigma_l^2(t_k)\right|\right\} = 0$, for $l = 1,\cdots,d$,
and
[III] $\lim_{h\downarrow 0}\limsup_{n\longrightarrow\infty}\frac{1}{h}E\left\{\left|E\left\{\left(X_n^{l_1}(t_k+h)-X_n^{l_1}(t_k)\right)\left(X_n^{l_2}(t_k+h)-X_n^{l_2}(t_k)\right)\right|\right.\right.$
$\left.\left.\left|X_n(t_1),X_n(t_2),\cdots,X_n(t_k)\right\}\right|\right\} = 0$, for $l_1 \neq l_2$.

Let us verify (I): Fix $0 < t < 1$, for reasons of convenience, we put $m_1 := [k_nt]$, and $m_2 := [k_n(t+h)] - [k_nt]$, where $0 < h \leqslant 1-t$. Consequently, for large n we have $1 \leqslant m_1 < m_1+m_2 \leqslant k_n$.
Suppose that we know the values of $\xi_{n1},\cdots,\xi_{nm_1}$, then $\xi_{n,m_1+1},\cdots,\xi_{n,m_1+m_2}$ are conditionally distributed as a sample of points of size m_2 taken from the population $x_{n1},x_{n2},\cdots,x_{nk_n}$ after removing the points corresponding to the sample $\xi_{n1},\cdots,\xi_{nm_1}$. Also, the values of $\varepsilon_{n,m_1+1},\cdots,\varepsilon_{n,m_1+m_2}$ are taken independently of each other, and also independently of the values of $\varepsilon_{n1},\cdots,\varepsilon_{nm_1}$. And consequently, we have similarly to the beginning of the proof:
$E\left\{\sum_{i=m_1+1}^{m_1+m_2}\zeta_{ni}\middle|\zeta_{n1},\cdots,\zeta_{nm_1}\right\} = 0$, which implies

$E\left\{X_n(t+h)-X_n(t)\middle|\zeta_{n1},\cdots,\zeta_{n[k_nt]}\right\}=0.$
Now, for $t=t_k$, and since

$$\sigma(X_n(t_1),\cdots,X_n(t_k))\subseteq\sigma(\zeta_{n1},\cdots,\zeta_{n[k_nt_k]}),$$

we have,
$E\left\{X_n\left(t_k+h\right)-X_n\left(t_k\right)\middle|X_n\left(t_1\right),\cdots,X_n\left(t_k\right)\right\}=0.$
And for $l=1,\cdots,d$ we find
$\frac{1}{h}E\left\{\left|E\left\{X_n^l(t_k+h)-X_n^l(t_k)\middle|X_n(t_1),\cdots,X_n(t_k)\right\}-h\cdot 0\cdot X_n^l(t_k)\right|\right\}=0.$
Hence, (I) is verified clearly.
Also, for $l=1,\cdots,d$, we have
$E\left\{\left(\sum\limits_{i=m_1+1}^{m_1+m_2}\zeta_{ni}^l\right)^2\middle|\zeta_{n1},\cdots,\zeta_{nm_1}\right\}=$
$=m_2E\left\{\left(\zeta_{n,m_1+1}^l\right)^2\middle|\zeta_{n1},\cdots,\zeta_{nm_1}\right\}+$
$+m_2(m_2-1)E\left\{\zeta_{n,m_1+1}^l\zeta_{n,m_1+2}^l\middle|\zeta_{n1},\cdots,\zeta_{nm_1}\right\}$

$=\frac{m_2}{(k_n-m_1)}\left(1-\sum\limits_{i=1}^{m_1}(\zeta_{ni}^l)^2\right)+0.$
Therefore, for $t=t_k$ we find
$E\left\{\left(X_n^l(t_k+h)-X_n^l(t_k)\right)^2\middle|\zeta_{n1},\cdots,\zeta_{n[k_nt_k]}\right\}=B_n(h)\left(1-\sum\limits_{i=1}^{[k_nt_k]}(\zeta_{ni}^l)^2\right),$
where

$$B_n(h):=\frac{[k_n(t_k+h)]-[k_nt_k]}{k_n-[k_nt_k]}.$$

Since

$$\sigma(X_n(t_1),\cdots,X_n(t_k))\subseteq\sigma(\zeta_{n1},\cdots,\zeta_{n[k_nt_k]}),$$

and from chapter one in this dissertation we have

$$\mathcal{C}_1\subseteq\mathcal{C}_2\Longrightarrow E\left\{|E(f|\mathcal{C}_1)|\right\}\leqslant E\left\{|E(f|\mathcal{C}_2)|\right\}.$$

So, we can write here
$\frac{1}{h}E\left\{\left|E\left\{\left(X_n^l(t_k+h)-X_n^l(t_k)\right)^2\middle|X_n(t_1),\cdots,X_n(t_k)\right\}-h\cdot 1\right|\right\}\leqslant$
$\leqslant\frac{1}{h}E\left\{\left|E\left\{\left(X_n^l(t_k+h)-X_n^l(t_k)\right)^2\middle|\zeta_{n1},\cdots,\zeta_{n[k_nt_k]}\right\}-h\cdot 1\right|\right\}\leqslant$
$\leqslant\frac{|B_n(h)|}{h}E\left\{\left|\sum\limits_{i=1}^{[k_nt_k]}(\zeta_{ni}^l)^2-\frac{[k_nt_k]}{k_n}\right|\right\}+\left|\frac{B_n(h)}{h}\left(1-\frac{[k_nt_k]}{k_n}\right)-1\right|$
But also we see that

$$B_n(h)\underset{n\longrightarrow\infty}{\longrightarrow}\frac{h}{1-t_k},$$

and

$$\left|\frac{B_n(h)}{h}\left(1-\frac{[k_nt_k]}{k_n}\right)-1\right| \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

So, to verify (II) it remains to prove the validity of the limit

$$E\left\{\left|\sum_{i=1}^{[k_nt_k]}(\zeta_{ni}^l)^2-\frac{[k_nt_k]}{k_n}\right|\right\} \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

For this, and from the beginning of the proof we can write that

$$E\left(\sum_{i=1}^{[k_nt_k]}\left(\zeta_{ni}^l\right)^2\right)=\frac{[k_nt_k]}{k_n} \underset{n\longrightarrow\infty}{\longrightarrow} t_k,$$

and let us here compute the second moment of $\sum\limits_{i=1}^{[k_nt_k]}\left(\zeta_{ni}^l\right)^2$.

$$E\left(\sum_{i=1}^{[k_nt_k]}\left(\zeta_{ni}^l\right)^2\right)^2=\frac{[k_nt_k](k_n-[k_nt_k])}{k_n(k_n-1)}\underbrace{\sum_{i=1}^{k_n}(x_{ni}^l)^4}_{\underset{n\longrightarrow\infty}{\longrightarrow}0}+\frac{[k_nt_k]([k_nt_k]-1)}{k_n(k_n-1)} \underset{n\longrightarrow\infty}{\longrightarrow} t_k^2.$$

But this implies $Var\left(\sum\limits_{i=1}^{[k_nt_k]}\left(\zeta_{ni}^l\right)^2\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0.$

And consequently, the limit

$$E\left\{\left|\sum_{i=1}^{[k_nt_k]}(\zeta_{ni}^l)^2-\frac{[k_nt_k]}{k_n}\right|\right\} \underset{n\longrightarrow\infty}{\longrightarrow} 0$$

is valid indeed.

Therefore, also the condition (II) is verified.

Now, let us verify the condition (III).

For $1\leqslant l_1\neq l_2\leqslant d$, we have

$$E\left\{\left(\sum_{i=m_1+1}^{m_1+m_2}\zeta_{ni}^{l_1}\right)\left(\sum_{i=m_1+1}^{m_1+m_2}\zeta_{ni}^{l_2}\right)\Big|\zeta_{n1},\cdots,\zeta_{nm_1}\right\}=$$

$$=m_2E\left\{\zeta_{n,m_1+1}^{l_1}\zeta_{n,m_1+1}^{l_2}\Big|\zeta_{n1},\cdots,\zeta_{nm_1}\right\}+$$
$$+m_2(m_2-1)E\left\{\zeta_{n,m_1+1}^{l_1}\zeta_{n,m_1+2}^{l_2}\Big|\zeta_{n1},\cdots,\zeta_{nm_1}\right\}$$

$$=-\frac{m_2}{k_n-m_1}\left(\sum_{i=1}^{m_1}\xi_{ni}^{l_1}\xi_{ni}^{l_2}\right)+0.$$

Therefore, for $t=t_k$ we find

$$E\left\{\left(X_n^{l_1}(t_k+h)-X_n^{l_1}(t_k)\right)\left(X_n^{l_2}(t_k+h)-X_n^{l_2}(t_k)\right)\Big|\zeta_{n1},\cdots,\zeta_{n[k_nt_k]}\right\}=$$

$$= -B_n(h)\left(\sum_{i=1}^{[k_nt_k]} \zeta_{ni}^{l_1}\zeta_{ni}^{l_2}\right),$$

$\frac{1}{h}E\left\{\left|E\left\{\left(X_n^{l_1}(t_k+h)-X_n^{l_1}(t_k)\right)\left(X_n^{l_2}(t_k+h)-X_n^{l_2}(t_k)\right)\big|\zeta_{n1},\cdots,\zeta_{n[k_nt_k]}\right\}\right|\right\}$
$\leqslant \frac{|B_n(h)|}{h}E\left(\left|\sum_{i=1}^{[k_nt_k]} \zeta_{ni}^{l_1}\zeta_{ni}^{l_2}\right|\right)$

So, to verify (III) it remains to prove the validity of the limit

$$E\left(\left|\sum_{i=1}^{[k_nt_k]} \zeta_{ni}^{l_1}\zeta_{ni}^{l_2}\right|\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

For this, let us compute the moment of $\sum_{i=1}^{[k_nt_k]} \zeta_{ni}^{l_1}\zeta_{ni}^{l_2}$

$E\left(\sum_{i=1}^{[k_nt_k]} \zeta_{ni}^{l_1}\zeta_{ni}^{l_2}\right)^2 = \frac{[k_nt_k]}{k_n}\underbrace{\sum_{i=1}^{k_n}(x_{ni}^{l_1})^2(x_{ni}^{l_2})^2}_{\underset{n\longrightarrow\infty}{\longrightarrow}0}.$

And hence, we have verified the condition (1^o).
Now, we turn to the second part of the proof. So, let us verifiy the remark, mentioned at the beginning of the proof:
Since the maximum jump $\max_{1\leqslant i\leqslant k_n} \parallel x_{ni} \parallel$ in X_n tends to zero and $X_n(0) = 0, \forall n\in\mathbb{N}$, the mentioned remark is verified obviously.
Now, we turn to condition(3^o) which states:
There is a constant K such that, for all $l=1,\cdots,d$, and for all $t_1,t,t_2\in[0,1]$ satisfying $t_1\leqslant t\leqslant t_2$, then
$\limsup_{n\longrightarrow\infty} E\left\{\left(X_n^l(t)-X_n^l(t_1)\right)^2\left(X_n^l(t_2)-X_n^l(t)\right)^2\right\}\leqslant K(t_2-t_1)^2.$
First, if $t_2-t_1\leqslant\frac{1}{k_n}$, then $E\left\{\left(X_n^l(t)-X_n^l(t_1)\right)^2\left(X_n^l(t_2)-X_n^l(t)\right)^2\right\}=0$, and the condition holds.
Now, if $t_2-t_1>\frac{1}{k_n}$, then we find easily that:

$$E\left\{\left(X_n^l(t)-X_n^l(t_1)\right)^2\left(X_n^l(t_2)-X_n^l(t)\right)^2\right\} = \sum\{\zeta_{ni}^l\zeta_{nj}^l\zeta_{nk}^l\zeta_{nl}^l\},$$

where $[k_nt_1] < i,j \leqslant [k_nt]$, and $[k_nt] < k,l \leqslant [k_nt_2]$. Let us put here $m_1 := [k_nt]-[k_nt_1]$ and $m_2 := [k_nt_2]-[k_nt]$. We can write for n large enough
$Q_n^l := \sum\{\zeta_{ni}^l\zeta_{nj}^l\zeta_{nk}^l\zeta_{nl}^l\} =$
$= m_1m_2E\{\left(\zeta_{n1}^l\right)^2\left(\zeta_{n2}^l\right)^2\} + m_1m_2(m_1+m_2-2)E\{\left(\zeta_{n1}^l\right)^2\zeta_{n2}^l\zeta_{n3}^l\}+$
$+m_1(m_1-1)m_2(m_2-1)E\{\zeta_{n1}^l\zeta_{n2}^l\zeta_{n3}^l\zeta_{n4}^l\}.$

$= m_1 m_2 \frac{1-\tau_n^l}{k_n(k_n-1)} + 0$, where $\tau_n^l := \sum\limits_{i=1}^{k_n} (x_{ni}^l)^4 \leqslant 1$.
$\limsup\limits_{n \longrightarrow \infty} Q_n^l \leqslant \frac{1}{4} (t_2 - t_1)^2$.
This will lead to the validity of the limit

$$\limsup_{n \longrightarrow \infty} E\left\{ \left(X_n^l(t) - X_n^l(t_1)\right)^2 \left(X_n^l(t_2) - X_n^l(t)\right)^2 \right\} \leqslant \frac{1}{4} (t_2 - t_1)^2 .$$

Therefore, also the condition (3^o) is verified at least for $K \geqslant \frac{1}{4}$, and consequently the proof is complete. □

Remark 2.3.4. Under the same hypotheses of theorem 2.3.3, but here the sequence $\left(\sum_{i=1}^{k_n} x_{ni} x_{ni}^t\right)_{n \in \mathbb{N}}$ is bounded, and the array $\underline{x}_{k_n}$ is infinitesimal, also the matrices $s^2\left(\underline{x}_{k_n}\right)$ are assumed to be positive definite, then we have $X_n \overset{w}{\sim} \sqrt{\sum_{i=1}^{k_n} x_{ni} x_{ni}^t} W \ (\mathcal{C}_n)$. Or more shortly,
$X_n \overset{w}{\sim} \sqrt{s^2\left(\underline{x}_{k_n}\right)} W \ (\mathcal{C}_n)$. Where $s^2\left(\underline{x}_{k_n}\right) := \sum_{i=1}^{k_n} x_{ni} x_{ni}^t$. And where the assertion means by definition

$$E(\varphi(X_n)) - E\left(\varphi\left(\sqrt{s^2\left(\underline{x}_{k_n}\right)} W\right)\right) \longrightarrow 0,$$

or more explicitly

$$\frac{1}{2^{k_n} k_n!} \sum_{\substack{\pi \\ \varepsilon_1, \cdots, \varepsilon_{k_n} \in \{-1,+1\}}} \varphi\left(\sum_{i=1}^{[k_n(\cdot)]} \varepsilon_i x_{n\pi(i)}\right) - \int\limits_{(C[0,1])^d} \varphi\left(\left(\sqrt{s^2(\underline{x}_{k_n})}\right) W\right) d\mu \longrightarrow 0.$$

For all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where μ is the the Wiener measure, defined $(C[0,1])^d$, and where π here ranging over all bijective functions, defined on $\{1, 2, \cdots, k_n\}$.

Proof: It is sufficient to prove it in the case where there exists $\varepsilon > 0$, such that $\| s^2(\underline{x}_{k_n}) \| > \varepsilon$ is valid for all $n \in \mathbb{N}$. We can reduce the case of this remark to the case of theorem 2.3.3. For this and since that for each $n \in \mathbb{N}$ the matrix $\sum_{i=1}^{k_n} x_{ni} x_{ni}^t$ is positive definite, we put $y_{ni} := \left(\sum_{i=1}^{k_n} x_{ni} x_{ni}^t\right)^{-\frac{1}{2}} x_{ni}$, $i = 1, \cdots, k_n$, $n \in \mathbb{N}$, then we have
$\left(\sum_{i=1}^{k_n} x_{ni} x_{ni}^t\right)^{-\frac{1}{2}} X_n \xrightarrow{\mathcal{D}} W$. Since the sequence $\left(\sum_{i=1}^{k_n} x_{ni} x_{ni}^t\right)_{n \in \mathbb{N}}$ is bounded, the assertion is valid. □

The following theorem is intended to be used later when we shall deal with permutation statistics for the hypothesis of independence H_2.

Theorem 2.3.5. For each $n \in \mathbb{N}$, let x_{n1}, x_{n2}, $\cdots$, x_{nk_n}, be a sequence of d_1-dimensional arrays of real numbers which are not distinct necessarily, and let y_{n1}, y_{n2}, $\cdots$, y_{nk_n}, be a sequence of d_2-dimensional arrays of real numbers which are not distinct necessarily also. let further $(\xi_{n1}, \xi_{n2}, \cdots, \xi_{nk_n})$, and $(\eta_{n1}, \eta_{n2}, \cdots, \eta_{nk_n})$ be independent arrays of random arrays, and taking values in the set of all the permutations of $(x_{n1}, x_{n2}, \cdots, x_{nk_n})$, and the set of all the permutations of $(y_{n1}, y_{n2}, \cdots, y_{nk_n})$ respectively, which are $\{\pi(x_{n1}, x_{n2}, \cdots, x_{nk_n}) : \pi \in \Pi_{k_n}\}$, and $\{\pi(y_{n1}, y_{n2}, \cdots, y_{nk_n}) : \pi \in \Pi_{k_n}\}$ respectively, where Π_{k_n} is the symmetric group of the permutation functions of order k_n, and we suppose further that each one of these permutations having probability $1/k_n!$. Finally, let us define the random arrays $\zeta_{n1}, \zeta_{n2}, \cdots, \zeta_{nk_n}$ by $\zeta_{ni} := \xi_{ni}\eta^t_{ni}, i = 1, \cdots, k_n$. And let us define X_n by $X_n(t) := \sum_{i=1}^{[k_n t]} \zeta_{ni}$, $X_n(t) := 0$, whenever $0 \leqslant t < 1/k_n$, where $[\cdot]$ stands for the integral value function. Then we assert that:
If $\frac{1}{\sqrt{k_n}} \sum\limits_{i=1}^{k_n} x_{ni}x^t_{ni} = I_{d_1}$, $\frac{1}{\sqrt{k_n}} \sum\limits_{i=1}^{k_n} y_{ni}y^t_{ni} = I_{d_2}$, $\sum\limits_{i=1}^{k_n} x_{ni} = 0$, $\sum\limits_{i=1}^{k_n} y_{ni} = 0$, and $\max_{1\leqslant i\leqslant k_n} \parallel x_{ni} \parallel \underset{n\longrightarrow\infty}{\longrightarrow} 0$, $\max_{1\leqslant i\leqslant k_n} \parallel y_{ni} \parallel \underset{n\longrightarrow\infty}{\longrightarrow} 0$, then we have

$$X_n \overset{\mathcal{D}}{\longrightarrow} W,$$

where W here is a $d_1 \times d_2$-dimensional Brownian motion.

Proof: We shall begin with some preliminary computations:

$$\forall j = 1, \cdots, d_1,\ \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} (x^j_{ni})^4 \leqslant \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} (x^j_{ni})^2 \left(\max_{1\leqslant i\leqslant k_n} |x^j_{ni}| \right)^2 \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

$$\forall j = 1, \cdots, d_2,\ \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} (y^j_{ni})^4 \leqslant \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} (y^j_{ni})^2 \left(\max_{1\leqslant i\leqslant k_n} |y^j_{ni}| \right)^2 \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

$$\sum_{i=1}^{k_n} x^{j_1}_{ni} x^{j_2}_{ni} = \sum_{i=1}^{k_n} y^{j_1}_{ni} y^{j_2}_{ni} = \begin{cases} \sqrt{k_n} & : j_1 = j_2 \\ 0 & : j_1 \neq j_2 \end{cases},$$

$$\sum_{1\leqslant i_1 \neq i_2 \leqslant k_n} x^{j_1}_{ni_1} x^{j_2}_{ni_2} = \sum_{1\leqslant i_1 \neq i_2 \leqslant k_n} y^{j_1}_{ni_1} y^{j_2}_{ni_2} = \begin{cases} -\sqrt{k_n} & : j_1 = j_2 \\ 0 & : j_1 \neq j_2 \end{cases},$$

and therefore,

$$E(\xi_{ni_1}^{j_1}\xi_{ni_2}^{j_2}) = E(\eta_{ni_1}^{j_1}\eta_{ni_2}^{j_2}) = \begin{cases} \frac{\sqrt{k_n}}{k_n} & : j_1 = j_2, i_1 = i_2 \\ 0 & : j_1 \neq j_2, i_1 = i_2 \\ -\frac{\sqrt{k_n}}{k_n(k_n-1)} & : j_1 = j_2, i_1 \neq i_2 \\ 0 & : j_1 \neq j_2, i_1 \neq i_2 \end{cases}.$$

$E\left(X_n^{j_1j_2}(t)\right) = E\left(\sum_{i=1}^{[k_nt]} \xi_{ni}^{j_1}\eta_{ni}^{j_2}\right) = \sum_{i=1}^{[k_nt]} E\left(\xi_{ni}^{j_1}\right) E\left(\eta_{ni}^{j_2}\right) = 0,$
and if $j_1 \neq j_3$, or $j_2 \neq j_4$, then
$E\left(X_n^{j_1j_2}(t)X_n^{j_3j_4}(t)\right) = E\left(\sum_{i_1=1}^{[k_nt]}\sum_{i_2=1}^{[k_nt]} \xi_{ni_1}^{j_1}\xi_{ni_2}^{j_3}\eta_{ni_1}^{j_2}\eta_{ni_2}^{j_4}\right) = 0.$
Also, if $j_1 = j_3$, and $j_2 = j_4$, then
$E\left(X_n^{j_1j_2}(t)\right)^2 = E\left(\sum_{i_1=1}^{[k_nt]}\sum_{i_2=1}^{[k_nt]} \xi_{ni_1}^{j_1}\xi_{ni_2}^{j_1}\eta_{ni_1}^{j_2}\eta_{ni_2}^{j_2}\right) =$
$= E\left(\sum_{i=1}^{[k_nt]} \xi_{ni}^{j_1}\xi_{ni}^{j_1}\eta_{ni}^{j_2}\eta_{ni}^{j_2}\right) + E\left(\sum_{i_1\neq i_2} \xi_{ni_1}^{j_1}\xi_{ni_2}^{j_1}\eta_{ni_1}^{j_2}\eta_{ni_2}^{j_2}\right)$
$= \underbrace{\sum_i E\left((\xi_{ni}^{j_1})^2\right) E\left((\eta_{ni}^{j_2})^2\right)}_{=\frac{[k_nt]}{k_n} \underset{n\longrightarrow\infty}{\longrightarrow} t} + \underbrace{\sum_{i_1\neq i_2} E\left(\xi_{ni_1}^{j_1}\xi_{ni_2}^{j_1}\eta_{ni_1}^{j_2}\eta_{ni_2}^{j_2}\right)}_{=\frac{[k_nt]([k_nt]-1)}{k_n(k_n-1)^2} \underset{n\longrightarrow\infty}{\longrightarrow} 0}.$

So far, we have found

$$E\left(X_n^{j_1j_2}(t)\right) = 0,$$

$$E\left(X_n^{j_1j_2}(t)\right)^2 \underset{n\longrightarrow\infty}{\longrightarrow} t,$$

and

$$E\left(X_n^{j_1j_2}(t)X_n^{j_3j_4}(t)\right) = 0, \text{ if } j_1 \neq j_3 \text{ or } j_2 \neq j_4.$$

We shall take here similar steps to those of the proof of theorem 2.3.3. Now, for $j_1 = 1, \cdots d_1$, $j_2 = 1, \cdots d_2$, we put $\rho_{j_1j_2}(t) = 0$, and $\sigma_{j_1j_2}^2(t) = 1$, we begin with verifying the condition (1^o), which states here:
[1^o] If $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k < 1$, then
[I] $\lim_{h\downarrow 0}\limsup_{n\longrightarrow\infty}\frac{1}{h}E\left\{\left|E\left\{X_n^{j_1j_2}(t_k+h) - X_n^{j_1j_2}(t_k)\middle| X_n(t_1), X_n(t_2), \cdots\right.\right.\right.$
$\left.\left.\left.\cdots, X_n(t_k)\right\} - h\rho_{j_1j_2}(t_k)X_n^{j_1j_2}(t_k)\right|\right\} = 0,$
for $j_1 = 1, \cdots, d_1$, and $j_2 = 1, \cdots, d_2$,
[II] $\lim_{h\downarrow 0}\limsup_{n\longrightarrow\infty}\frac{1}{h}E\left\{\left|E\left\{\left(X_n^{j_1j_2}(t_k+h) - X_n^{j_1j_2}(t_k)\right)^2 \middle| X_n(t_1), X_n(t_2), \cdots\right.\right.\right.$

$$\cdots, X_n(t_k)\} - h\sigma^2_{j_1j_2}(t_k)\Big|\Big\} = 0,$$

for $j_1 = 1, \cdots, d_1$, and $j_2 = 1, \cdots, d_2$, and

[III] $\lim_{h\downarrow 0} \limsup_{n\longrightarrow\infty} \frac{1}{h} E\left\{\Big| E\Big\{ (X_n^{j_1j_2}(t_k+h) - X_n^{j_1j_2}(t_k))\right.$

$$\left.(X_n^{j_3j_4}(t_k+h) - X_n^{j_3j_4}(t_k)) \,\big| X_n(t_1), X_n(t_2), \cdots, X_n(t_k)\Big\}\Big|\right\} = 0,$$

for $j_1, j_3 = 1, \cdots, d_1$, and for $j_2, j_4 = 1, \cdots, d_2$, and only if either $j_1 \neq j_3$ or $j_2 \neq j_4$ is valid.

Let us start with verifying the part (I): Fix $0 < t < 1$, let us here put $m_1 := [k_nt]$, $m_2 := [k_n(t+h)] - [k_nt]$, where $0 < h \leqslant 1-t$. Consequently, for large n we have $1 \leqslant m_1 < m_1 + m_2 \leqslant k_n$.

Suppose that we know the values of $\xi_{n1}, \cdots, \xi_{nm_1}$, then $\xi_{n,m_1+1}, \cdots, \xi_{n,m_1+m_2}$ are conditionally distributed as a sample of points of size m_2 taken from the population $x_{n1}, x_{n2}, \cdots, x_{nk_n}$ after removing the points corresponding to the sample $\xi_{n1}, \cdots, \xi_{nm_1}$. Suppose similarly, that we know the values of $\eta_{n1}, \cdots, \eta_{nm_1}$, then $\eta_{n,m_1+1}, \cdots, \eta_{n,m_1+m_2}$ are conditionally distributed as a sample of points of size m_2 taken from the population $y_{n1}, y_{n2}, \cdots, y_{nk_n}$ after removing the points corresponding to the sample $\eta_{n1}, \cdots, \eta_{nm_1}$.

And we have similarly to the preliminary computations at the beginning of this proof:

$$E\left\{\sum_{i=m_1+1}^{m_1+m_2} \xi_{ni}^{j_1}\eta_{ni}^{j_2}\middle| \xi_{n1}, \cdots, \xi_{nm_1}; \eta_{n1}, \cdots, \eta_{nm_1}\right\} =$$
$$= \sum_{i=m_1+1}^{m_1+m_2} E\left\{\xi_{ni}^{j_1}\eta_{ni}^{j_2}\middle| \xi_{n1}, \cdots, \xi_{nm_1}; \eta_{n1}, \cdots, \eta_{nm_1}\right\}$$
$$= \sum_{i=m_1+1}^{m_1+m_2} \frac{\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\right)\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\right)}{(k_n-m_1)^2}$$
$$= \frac{[k_n(t+h)]-[k_nt]}{(k_n-[k_nt])^2}\left(\sum_{l=1}^{[k_nt]}\xi_{nl}^{j_1}\right)\left(\sum_{l=1}^{[k_nt]}\eta_{nl}^{j_2}\right).$$

But on the other hand we have:

$$E\left(\frac{[k_n(t+h)]-[k_nt]}{(k_n-[k_nt])^2}\left(\sum_{l=1}^{[k_nt]}\xi_{nl}^{j_1}\right)\left(\sum_{l=1}^{[k_nt]}\eta_{nl}^{j_2}\right)\right)^2 =$$
$$= \left(\frac{[k_n(t+h)]-[k_nt]}{(k_n-[k_nt])^2}\right)^2 E\left(\sum_{l=1}^{[k_nt]}\xi_{nl}^{j_1}\right)^2 E\left(\sum_{l=1}^{[k_nt]}\eta_{nl}^{j_2}\right)^2$$
$$= \left(\frac{[k_n(t+h)]-[k_nt]}{(k_n-[k_nt])^2}\right)^2 \left(\frac{[k_nt]}{k_n} - \frac{[k_nt]([k_nt]-1)}{k_n(k_n-1)}\right)^2 k_n \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

Therefore, for $t = t_k$, and since

$$\sigma(X_n(t_1), \cdots, X_n(t_k)) \subseteq \sigma(\xi_{n1}, \cdots, \xi_{n[k_nt_k]}; \eta_{n1}, \cdots, \eta_{n[k_nt_k]}),$$

we conclude that

$$\frac{1}{h}E\left\{\left|E\left\{X_n^{j_1j_2}(t_k+h)-X_n^{j_1j_2}(t_k)\middle|X_n(t_1),X_n(t_2),\cdots\right.\right.\right.$$

$$\left.\left.\left.\cdots,X_n(t_k)\right\}-h\cdot 0\cdot X_n^{j_1j_2}(t_k)\right|\right\},$$

for $j_1=1,\cdots,d_1$, and $j_2=1,\cdots,d_2$, tend to zero with n goes to infinity.
Hence, the part (I) of the condition (1^o) is verified indeed.
We turn now to the part (II) of the condition (1^o).
We prefer here to make some necessary preliminary computations.
Fix $0<t<1$, again here we put $m_1:=[k_nt]$, and $m_2:=[k_n(t+h)]-[k_nt]$.
Also, if $j_1=j_2$ we put $j:=j_1=j_2$, in the following formulas below, which are similar to those formulas at the beginning of the proof:

$$\sum_{i=m_1+1}^{k_n}x_{ni}^{j_1}x_{ni}^{j_2}=\begin{cases}\sqrt{k_n}-\sum\limits_{l=1}^{m_1}\left(x_{nl}^{j}\right)^2 & :j_1=j_2\\ -\sum\limits_{l=1}^{m_1}x_{nl}^{j_1}x_{nl}^{j_2} & :j_1\neq j_2\end{cases},$$

$$\sum_{m_1+1\leqslant i_1\neq i_2\leqslant k_n}x_{ni_1}^{j_1}x_{ni_2}^{j_2}=\begin{cases}\left(\sum\limits_{l=1}^{m_1}x_{nl}^{j}\right)^2+\sum\limits_{l=1}^{m_1}\left(x_{nl}^{j}\right)^2-\sqrt{k_n} & :j_1=j_2\\ \left(\sum\limits_{l=1}^{m_1}x_{nl}^{j_1}\right)\left(\sum\limits_{l=1}^{m_1}x_{nl}^{j_2}\right)+\sum\limits_{l=1}^{m_1}x_{nl}^{j_1}x_{nl}^{j_2} & :j_1\neq j_2\end{cases},$$

and similar formulas are valid when we put the y's in place of x's above.
Therefore, we have for $i_1,\ i_2>m_1+1$

$$E\left(\xi_{ni_1}^{j_1}\xi_{ni_2}^{j_1}\eta_{ni_1}^{j_2}\eta_{ni_2}^{j_2}\middle|\xi_{n1},\cdots,\xi_{nm_1};\eta_{n1},\cdots,\eta_{nm_1}\right)=$$

$$=\begin{cases}\dfrac{\left(\sqrt{k_n}-\sum\limits_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2\right)\left(\sqrt{k_n}-\sum\limits_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2\right)}{(k_n-m_1)^2} & :i_1=i_2\\ \dfrac{\left(\left(\sum\limits_{l=1}^{m_1}\xi_{nl}^{j_1}\right)^2+\sum\limits_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2-\sqrt{k_n}\right)\left(\left(\sum\limits_{l=1}^{m_1}\eta_{nl}^{j_2}\right)^2+\sum\limits_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2-\sqrt{k_n}\right)}{(k_n-m_1)^2(k_n-m_1-1)^2} & :i_1\neq i_2\end{cases}.$$

Also, for fixed $j_1,\ j_2$ we find

$$E\left\{\left(\sum_{i=m_1+1}^{m_1+m_2}\xi_{ni}^{j_1}\eta_{ni}^{j_2}\right)^2\middle|\xi_{n1},\cdots,\xi_{nm_1};\eta_{n1},\cdots,\eta_{nm_1}\right\}=$$

$$=m_2\frac{\left(\sqrt{k_n}-\sum\limits_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2\right)\left(\sqrt{k_n}-\sum\limits_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2\right)}{(k_n-m_1)^2}+$$

$$+m_2(m_2-1)\frac{\left(\left(\sum\limits_{l=1}^{m_1}\xi_{nl}^{j_1}\right)^2+\sum\limits_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2-\sqrt{k_n}\right)\left(\left(\sum\limits_{l=1}^{m_1}\eta_{nl}^{j_2}\right)^2+\sum\limits_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2-\sqrt{k_n}\right)}{(k_n-m_1)^2(k_n-m_1-1)^2}.$$

For notational convenience, we put

$$Q_{n1}:=m_2\frac{\left(\sqrt{k_n}-\sum\limits_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2\right)\left(\sqrt{k_n}-\sum\limits_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2\right)}{(k_n-m_1)^2},$$

$$Q_{n2} := m_2(m_2-1)\frac{\left(\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\right)^2+\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2-\sqrt{k_n}\right)\left(\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\right)^2+\sum_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2-\sqrt{k_n}\right)}{(k_n-m_1)^2(k_n-m_1-1)^2}.$$

And we shall now deal with each quantity of Q_{n1}, Q_{n2} alone.

$$Q_{n1} = \underbrace{\frac{m_2 k_n}{(k_n-m_1)^2}}_{\underset{n\longrightarrow\infty}{\longrightarrow}\frac{h}{(1-t)^2}}\left(1-\frac{1}{\sqrt{k_n}}\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2\right)\left(1-\frac{1}{\sqrt{k_n}}\sum_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2\right),$$

$$E\left(1-\frac{1}{\sqrt{k_n}}\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2\right) = 1-\frac{[k_n t]}{k_n}\underset{n\longrightarrow\infty}{\longrightarrow}(1-t),$$

$$E\left(1-\frac{1}{\sqrt{k_n}}\sum_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2\right) = 1-\frac{[k_n t]}{k_n}\underset{n\longrightarrow\infty}{\longrightarrow}(1-t),$$

$$E\left(1-\frac{1}{\sqrt{k_n}}\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2\right)^2 =$$

$$= E\left(1+\frac{1}{k_n}\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^4+\frac{1}{k_n}\sum_{l_1\neq l_2}\left(\xi_{nl_1}^{j_1}\right)^2\left(\xi_{nl_2}^{j_1}\right)^2-\frac{2}{\sqrt{k_n}}\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2\right)$$

$$= 1+\frac{[k_n t]}{(k_n)^2}\sum_{l=1}^{k_n}(x_{nl}^{j_1})^4+\frac{[k_n t]([k_n t]-1)}{(k_n)^2(k_n-1)}\left(\left(\sum_{l=1}^{k_n}(x_{nl}^{j_1})^2\right)^2-\sum_{l=1}^{k_n}(x_{nl}^{j_1})^4\right)-2\frac{[k_n t]}{k_n},$$

$$E\left(1-\frac{1}{\sqrt{k_n}}\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2\right)^2\underset{n\longrightarrow\infty}{\longrightarrow}(1-t)^2,$$

$$E\left(1-\frac{1}{\sqrt{k_n}}\sum_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2\right)^2\underset{n\longrightarrow\infty}{\longrightarrow}(1-t)^2,$$

$E\left(Q_{n1}\right)\underset{n\longrightarrow\infty}{\longrightarrow}h$, $E\left(Q_{n1}\right)^2\underset{n\longrightarrow\infty}{\longrightarrow}h^2$.

We turn now to deal Q_{n2}

$$Q_{n2} = \underbrace{\frac{m_2(m_2-1)}{(k_n-m_1)^2}}_{\underset{n\longrightarrow\infty}{\longrightarrow}\frac{h^2}{(1-t)^2}}\frac{\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\right)^2+\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2-\sqrt{k_n}}{(k_n-m_1-1)}\frac{\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\right)^2+\sum_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2-\sqrt{k_n}}{(k_n-m_1-1)}.$$

By routine and similar computations we easily find:

$$E\left(\frac{\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\right)^2+\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2-\sqrt{k_n}}{(k_n-m_1-1)}\right)^2\underset{n\longrightarrow\infty}{\longrightarrow}0,$$

$$E\left(\frac{\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\right)^2+\sum_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2-\sqrt{k_n}}{(k_n-m_1-1)}\right)^2\underset{n\longrightarrow\infty}{\longrightarrow}0,$$

$E\left(Q_{n2}\right)^2\underset{n\longrightarrow\infty}{\longrightarrow}0$.

Consequently, we have

$E\left(Q_{n1}+Q_{n2}\right)\underset{n\longrightarrow\infty}{\longrightarrow}h$,

$E\left(Q_{n1}+Q_{n2}\right)^2 \underset{n\longrightarrow\infty}{\longrightarrow} h^2.$
Therefore, also we have
$E\left(E\left\{\left(\sum_{i=m_1+1}^{m_1+m_2}\xi_{ni}^{j_1}\eta_{ni}^{j_2}\right)^2\middle|\xi_{n1},\cdots,\xi_{nm_1};\eta_{n1},\cdots,\eta_{nm_1}\right\}\right)\underset{n\longrightarrow\infty}{\longrightarrow} h,$
$E\left(E\left\{\left(\sum_{i=m_1+1}^{m_1+m_2}\xi_{ni}^{j_1}\eta_{ni}^{j_2}\right)^2\middle|\xi_{n1},\cdots,\xi_{nm_1};\eta_{n1},\cdots,\eta_{nm_1}\right\}\right)^2\underset{n\longrightarrow\infty}{\longrightarrow} h^2.$
Now, since

$$\sigma(X_n(t_1),\cdots,X_n(t_k))\subseteq\sigma(\xi_{n1},\cdots,\xi_{n[k_nt_k]};\eta_{n1},\cdots,\eta_{n[k_nt_k]}),$$

we conclude that the part (II) of the condition (1^o) is verified indeed.
We turn now to deal with the the part (III) of the condition (1^o).
Fix $0<t<1$, again here we put $m_1:=[k_nt]$, and $m_2:=[k_n(t+h)]-[k_nt]$.
Also, similarly here we have
$E\left(\xi_{ni_1}^{j_1}\xi_{ni_2}^{j_3}\eta_{ni_1}^{j_2}\eta_{ni_2}^{j_4}\middle|\xi_{n1},\cdots,\xi_{nm_1};\eta_{n1},\cdots,\eta_{nm_1}\right)=$

$$=\begin{cases}
-\frac{\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\xi_{nl}^{j_3}\right)\left(\sqrt{k_n}-\sum_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2\right)}{(k_n-m_1)^2} & : \text{ if } \begin{cases} i_1=i_2\\ j_1\neq j_3\\ j_2=j_4\end{cases}\\
-\frac{\left(\sqrt{k_n}-\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2\right)\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\eta_{nl}^{j_4}\right)}{(k_n-m_1)^2} & : \text{ if } \begin{cases} i_1=i_2\\ j_1=j_3\\ j_2\neq j_4\end{cases}\\
\frac{\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\xi_{nl}^{j_3}\right)\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\eta_{nl}^{j_4}\right)}{(k_n-m_1)^2} & : \text{ if } \begin{cases} i_1=i_2\\ j_1\neq j_3\\ j_2\neq j_4\end{cases}\\
\frac{\left(\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\right)\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_3}\right)+\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\xi_{nl}^{j_3}\right)\left(\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\right)^2+\sum_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2-\sqrt{k_n}\right)}{(k_n-m_1)^2(k_n-m_1-1)^2} & : \text{ if } \begin{cases} i_1\neq i_2\\ j_1\neq j_3\\ j_2=j_4\end{cases}\\
\frac{\left(\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\right)^2+\sum_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2-\sqrt{k_n}\right)\left(\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\right)\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_4}\right)+\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\eta_{nl}^{j_4}\right)}{(k_n-m_1)^2(k_n-m_1-1)^2} & : \text{ if } \begin{cases} i_1\neq i_2\\ j_1=j_3\\ j_2\neq j_4\end{cases}\\
\frac{\left(\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\right)\left(\sum_{l=1}^{m_1}\xi_{nl}^{j_3}\right)+\sum_{l=1}^{m_1}\xi_{nl}^{j_1}\xi_{nl}^{j_3}\right)\left(\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\right)\left(\sum_{l=1}^{m_1}\eta_{nl}^{j_4}\right)+\sum_{l=1}^{m_1}\eta_{nl}^{j_2}\eta_{nl}^{j_4}\right)}{(k_n-m_1)^2(k_n-m_1-1)^2} & : \text{ if } \begin{cases} i_1\neq i_2\\ j_1\neq j_3\\ j_2\neq j_4\end{cases}
\end{cases}.$$

Also, for fixed j_1, j_2, j_3, j_4, such that either $j_1\neq j_3$, or $j_2\neq j_4$, is valid, and then we find
$E\left\{\left(\sum_{i=m_1+1}^{m_1+m_2}\xi_{ni}^{j_1}\eta_{ni}^{j_2}\right)\left(\sum_{i=m_1+1}^{m_1+m_2}\xi_{ni}^{j_3}\eta_{ni}^{j_4}\right)\middle|\xi_{n1},\cdots,\xi_{nm_1};\eta_{n1},\cdots,\eta_{nm_1}\right\}=$

$$= \left\{ -m_2 \frac{\left(\sum\limits_{l=1}^{m_1} \xi_{nl}^{j_1}\xi_{nl}^{j_3}\right)\left(\sqrt{k_n} - \sum\limits_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2\right)}{(k_n-m_1)^2} + \right.$$

$$\left. + m_2(m_2-1) \frac{\left(\left(\sum\limits_{l=1}^{m_1} \xi_{nl}^{j_1}\right)\left(\sum\limits_{l=1}^{m_1} \xi_{nl}^{j_3}\right) + \sum\limits_{l=1}^{m_1} \xi_{nl}^{j_1}\xi_{nl}^{j_3}\right)\left(\left(\sum\limits_{l=1}^{m_1} \eta_{nl}^{j_2}\right)^2 + \sum\limits_{l=1}^{m_1}\left(\eta_{nl}^{j_2}\right)^2 - \sqrt{k_n}\right)}{(k_n-m_1)^2(k_n-m_1-1)^2} \right\}$$

if $\begin{Bmatrix} j_1 \neq j_3 \\ j_2 = j_4 \end{Bmatrix} +$

$$+ \left\{ -m_2 \frac{\left(\sqrt{k_n} - \sum\limits_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2\right)\left(\sum\limits_{l=1}^{m_1} \eta_{nl}^{j_2}\eta_{nl}^{j_4}\right)}{(k_n-m_1)^2} + \right.$$

$$\left. + m_2(m_2-1) \frac{\left(\left(\sum\limits_{l=1}^{m_1} \xi_{nl}^{j_1}\right)^2 + \sum\limits_{l=1}^{m_1}\left(\xi_{nl}^{j_1}\right)^2 - \sqrt{k_n}\right)\left(\left(\sum\limits_{l=1}^{m_1} \eta_{nl}^{j_2}\right)\left(\sum\limits_{l=1}^{m_1} \eta_{nl}^{j_4}\right) + \sum\limits_{l=1}^{m_1} \eta_{nl}^{j_2}\eta_{nl}^{j_4}\right)}{(k_n-m_1)^2(k_n-m_1-1)^2} \right\}$$

if $\begin{Bmatrix} j_1 = j_3 \\ j_2 \neq j_4 \end{Bmatrix} +$

$$+ \left\{ m_2 \frac{\left(\sum\limits_{l=1}^{m_1} \xi_{nl}^{j_1}\xi_{nl}^{j_3}\right)\left(\sum\limits_{l=1}^{m_1} \eta_{nl}^{j_2}\eta_{nl}^{j_4}\right)}{(k_n-m_1)^2} + \right.$$

$$\left. + m_2(m_2-1) \frac{\left(\left(\sum\limits_{l=1}^{m_1} \xi_{nl}^{j_1}\right)\left(\sum\limits_{l=1}^{m_1} \xi_{nl}^{j_3}\right) + \sum\limits_{l=1}^{m_1} \xi_{nl}^{j_1}\xi_{nl}^{j_3}\right)\left(\left(\sum\limits_{l=1}^{m_1} \eta_{nl}^{j_2}\right)\left(\sum\limits_{l=1}^{m_1} \eta_{nl}^{j_4}\right) + \sum\limits_{l=1}^{m_1} \eta_{nl}^{j_2}\eta_{nl}^{j_4}\right)}{(k_n-m_1)^2(k_n-m_1-1)^2} \right\}$$

if $\begin{Bmatrix} j_1 \neq j_3 \\ j_2 \neq j_4 \end{Bmatrix}$

By using routine computations similarly to what we did before with the condition (II) we conclude that the condition (III) is valid indeed.

Now, since the maximum jump in $X_n(t)$ tends to zero, and also $X_n(0) = 0$, $\forall n \in \mathbb{N}$, the remark following the proof of theorem 2.2.6 is also verified.

Now, we turn to condition(3^o) which states: There is a constant K such that, for all $j_1 = 1, \cdots, d_1$, $j_2 = 1, \cdots, d_2$, and for all $t_1, t, t_2 \in [0, 1]$ satisfying $t_1 \leqslant t \leqslant t_2$, then

$$\limsup_{n \longrightarrow \infty} E\left\{\left(X_n^{j_1j_2}(t) - X_n^{j_1j_2}(t_1)\right)^2\left(X_n^{j_1j_2}(t_2) - X_n^{j_1j_2}(t)\right)^2\right\} \leqslant K(t_2-t_1)^2.$$

First, if $t_2 - t_1 \leqslant \frac{1}{k_n}$, then for all $n \in \mathbb{N}$ we have

$E\left\{\left(X_n^{j_1j_2}(t) - X_n^{j_1j_2}(t_1)\right)^2\left(X_n^{j_1j_2}(t_2) - X_n^{j_1j_2}(t)\right)^2\right\} = 0$, and the condition (III) holds in this case.

Now, if $t_2 - t_1 > \frac{1}{k_n}$, then we find easily that:

$E\left\{\left(X_n^{j_1j_2}(t) - X_n^{j_1j_2}(t_1)\right)^2\left(X_n^{j_1j_2}(t_2) - X_n^{j_1j_2}(t)\right)^2\right\} =$

$$= \sum_{[k_nt_1]<l_1\neq l_2\leqslant[k_nt]} \sum_{[k_nt]<l_3\neq l_4\leqslant[k_nt_2]} E\left\{\xi^{j_1}_{nl_1}\xi^{j_1}_{nl_2}\xi^{j_1}_{nl_3}\xi^{j_1}_{nl_4}\right\} E\left\{\eta^{j_2}_{nl_1}\eta^{j_2}_{nl_2}\eta^{j_2}_{nl_3}\eta^{j_2}_{nl_4}\right\} +$$
$$+ \sum_{[k_nt_1]<l_1\neq l_2\leqslant[k_nt]} \sum_{[k_nt]<l_3\leqslant[k_nt_2]} E\left\{\xi^{j_1}_{nl_1}\xi^{j_1}_{nl_2}\left(\xi^{j_1}_{nl_3}\right)^2\right\} E\left\{\eta^{j_2}_{nl_1}\eta^{j_2}_{nl_2}\left(\eta^{j_2}_{nl_3}\right)^2\right\} +$$
$$+ \sum_{[k_nt_1]<l_1\leqslant[k_nt]} \sum_{[k_nt]<l_2\neq l_3\leqslant[k_nt_2]} E\left\{\left(\xi^{j_1}_{nl_1}\right)^2\xi^{j_1}_{nl_2}\xi^{j_1}_{nl_3}\right\} E\left\{\left(\eta^{j_2}_{nl_1}\right)^2\eta^{j_2}_{nl_2}\eta^{j_2}_{nl_3}\right\} +$$
$$+ \sum_{[k_nt_1]<l_1\leqslant[k_nt]} \sum_{[k_nt]<l_2\leqslant[k_nt_2]} E\left\{\left(\xi^{j_1}_{nl_1}\right)^2\left(\xi^{j_1}_{nl_2}\right)^2\right\} E\left\{\left(\eta^{j_2}_{nl_1}\right)^2\left(\eta^{j_2}_{nl_2}\right)^2\right\}.$$

For reasons of convenience, we put here
$m_1 := [k_nt] - [k_nt_1]$, and $m_2 := [k_nt_2] - [k_nt]$.

$$E\left\{\left(X^{j_1j_2}_n(t) - X^{j_1j_2}_n(t_1)\right)^2\left(X^{j_1j_2}_n(t_2) - X^{j_1j_2}_n(t)\right)^2\right\} =$$
$$= \frac{m_1(m_1-1)m_2(m_2-1)}{k_n(k_n-1)(k_n-2)(k_n-3)}\frac{S_{11}\cdot S_{12}}{k_n(k_n-1)(k_n-2)(k_n-3)} + \frac{m_1(m_1-1)m_2}{k_n(k_n-1)(k_n-2)}\frac{S_{21}\cdot S_{22}}{k_n(k_n-1)(k_n-2)} +$$
$$+\frac{m_1m_2(m_2-1)}{k_n(k_n-1)(k_n-2)}\frac{S_{31}\cdot S_{32}}{k_n(k_n-1)(k_n-2)} + \frac{m_1m_2}{k_n(k_n-1)}\frac{S_{41}\cdot S_{42}}{k_n(k_n-1)}.$$

Where
$S_{11} := \sum x^{j_1}_{nl_1}x^{j_1}_{nl_2}x^{j_1}_{nl_3}x^{j_1}_{nl_4}$,
for l_1, l_2, l_3, l_4 are distinct pairwise and ranging over $1, \cdots, k_n$,
$S_{12} := \sum y^{j_2}_{nl_1}y^{j_2}_{nl_2}y^{j_2}_{nl_3}y^{j_2}_{nl_4}$,
for l_1, l_2, l_3, l_4 are distinct pairwise and ranging over $1, \cdots, k_n$,
$S_{21} := \sum x^{j_1}_{nl_1}x^{j_1}_{nl_2}\left(x^{j_1}_{nl_3}\right)^2$,
for l_1, l_2, l_3 are distinct pairwise and ranging over $1, \cdots, k_n$,
$S_{22} := \sum y^{j_2}_{nl_1}y^{j_2}_{nl_2}\left(y^{j_2}_{nl_3}\right)^2$,
for l_1, l_2, l_3 are distinct pairwise and ranging over $1, \cdots, k_n$,
$S_{31} := \sum \left(x^{j_1}_{nl_1}\right)^2 x^{j_1}_{nl_2}x^{j_1}_{nl_3}$,
for l_1, l_2, l_3 are distinct pairwise and ranging over $1, \cdots, k_n$,
$S_{32} := \sum \left(y^{j_2}_{nl_1}\right)^2 y^{j_2}_{nl_2}y^{j_2}_{nl_3}$,
for l_1, l_2, l_3 are distinct pairwise and ranging over $1, \cdots, k_n$,
$S_{41} := \sum \left(x^{j_1}_{nl_1}\right)^2\left(x^{j_1}_{nl_2}\right)^2$,
for l_1, l_2 are distinct and ranging over $1, \cdots, k_n$,
$S_{42} := \sum \left(y^{j_2}_{nl_1}\right)^2\left(y^{j_2}_{nl_2}\right)^2$,
for l_1, l_2 are distinct and ranging over $1, \cdots, k_n$.

Therefore, there are constants K, K_1, K_2, K_3, and K_4 such that, for all $j_1 = 1, \cdots, d_1$, and $j_2 = 1, \cdots, d_2$, we have

$$E\left\{\left(X^{j_1j_2}_n(t) - X^{j_1j_2}_n(t_1)\right)^2\left(X^{j_1j_2}_n(t_2) - X^{j_1j_2}_n(t)\right)^2\right\} \leqslant$$
$$\leqslant \frac{m_1(m_1-1)m_2(m_2-1)}{k_n(k_n-1)(k_n-2)(k_n-3)}K_1 + \frac{m_1(m_1-1)m_2}{k_n(k_n-1)(k_n-2)}K_2 + \frac{m_1m_2(m_2-1)}{k_n(k_n-1)(k_n-2)}K_3 + \frac{m_1m_2}{k_n(k_n-1)}K_4$$
$$\leqslant K(t_2 - t_1)^2.$$

Hence, the condition (3^o) is verified indeed, and consequently the proof is complete. □

Corollary 2.3.6. Under the same hypotheses of theorem 2.3.5 let us define the random arrays $\zeta_{n1}, \zeta_{n2}, \cdots, \zeta_{nk_n}$ by the following formulas $\zeta_{ni} := \xi_{ni} * \eta_{ni}, i = 1, \cdots, k_n$, and where $*$ here is a binary operation defined by $x * y := (x^1y^1, \cdots, x^dy^d)^t$, $\forall x, y \in \mathbb{R}^d$. And let us define X_n by $X_n(t) := \sum_{i=1}^{[k_nt]} \zeta_{ni}$, $X_n(t) := 0$ whenever $0 \leqslant t < 1/k_n$, where $[\cdot]$ stands for the integral value function. Then we assert that:

$$X_n \xrightarrow{\mathcal{D}} W,$$

where W here is a d-dimensional Brownian motion with independent arguments, each argument is a one-dimensional Brownian motion. $\square$

Corollary 2.3.7. In the same situation of theorem 2.3.5, but here we put for each $n \in \mathbb{N}$, $\zeta_{ni} := \xi_{ni}^t \eta_{ni}$, $i = 1, \cdots, k_n$, then we obtain:

$$X_n \xrightarrow{\mathcal{D}} \sqrt{d}W,$$

or more explicitly

$$\left(\sum_{i=1}^{[k_nt]} \xi_{ni}^j \eta_{ni}^j \right)_{0 \leqslant t \leqslant 1} \xrightarrow{\mathcal{D}} \left(\sqrt{d}W \right)_{0 \leqslant t \leqslant 1},$$

where W here is a one-dimensional Brownian motion.

Proof: From the proof of theorem 2.3.5 we have

$$\sum_{j=1}^{d} u_j X_n^{jj} \xrightarrow{\mathcal{D}} \sqrt{\sum_{j=1}^{d} (u_j)^2} W,$$

Now, we just put $u_j = 1$, $j = 1, \cdots, d$, above. This implies the validity of this corollary. $\square$

Remark 2.3.8. We keep all hypotheses of theorem 2.3.5, but we make some changes in the conditions of the x's and the y's, where here we suppose that the following sequences:
$\left(\frac{1}{\sqrt{k_n}} s^2(\underline{x}_{k_n}) \right)_{n \in \mathbb{N}}$, $\left(\frac{1}{\sqrt{k_n}} s^2(\underline{y}_{k_n}) \right)_{n \in \mathbb{N}}$, $\left(\frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} x_{ni} \right)_{n \in \mathbb{N}}$, and $\left(\frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} y_{ni} \right)_{n \in \mathbb{N}}$
are bounded. Where,

$s^2(\underline{x}_{k_n}) := \sum_{i=1}^{k_n}(x_{ni} - \bar{x}_n)(x_{ni} - \bar{x}_n)^t,$
$s^2(\underline{y}_{k_n}) := \sum_{i=1}^{k_n}(y_{ni} - \bar{y}_n)(y_{ni} - \bar{y}_n)^t,$
where $\bar{x}_n := \frac{1}{k_n}\sum_{i=1}^{k_n} x_{ni}$, and $\bar{y}_n := \frac{1}{k_n}\sum_{i=1}^{k_n} y_{ni}$. We suppose further that the arrays $\underline{x}_{k_n}$, and $\underline{y}_{k_n}$ are infinitesimal and that the matrices $s^2(\underline{x}_{k_n})$, $s^2(\underline{y}_{k_n})$ are positive definite for all $n \in \mathbb{N}$.
Then the assertion becomes

$$X_n \overset{w}{\sim} \left(\frac{1}{\sqrt{k_n}}\sqrt{s^2(\underline{x}_{k_n})}\, W_t \sqrt{s^2(\underline{y}_{k_n})} + \frac{t}{k_n}\sum_{i=1}^{k_n} x_{ni}\sum_{i=1}^{k_n} y_{ni}^t\right)_{0\leqslant t\leqslant 1} \quad (\mathcal{C}_n),$$

which means by definition

$$E(\varphi(X_n)) - E\left(\varphi\left(\frac{1}{\sqrt{k_n}}\sqrt{s^2(\underline{x}_{k_n})}\, W \sqrt{s^2(\underline{y}_{k_n})} + \frac{(\cdot)}{k_n}\sum_{i=1}^{k_n} x_{ni}\sum_{i=1}^{k_n} y_{ni}^t\right)\right) \longrightarrow 0,$$

or more explicitly

$$\frac{1}{(k_n!)^2}\sum_{\pi_1,\pi_2}\varphi\left(\sum_{i=1}^{[k_n(\cdot)]} x_{n\pi_1(i)}y^t_{n\pi_2(i)}\right) -$$
$$- \int_{(C[0,1)]^{d_1\times d_2}}\varphi\left(\frac{1}{\sqrt{k_n}}\sqrt{s^2(\underline{x}_{k_n})}\, W \sqrt{s^2(\underline{y}_{k_n})} + \frac{(\cdot)}{k_n}\sum_{i=1}^{k_n} x_{ni}\sum_{i=1}^{k_n} y_{ni}^t\right) d\mu \longrightarrow 0.$$

For all bounded uniformly continuous functions $\varphi : (D[0,1])^{d_1\times d_2} \longrightarrow \mathbb{R}$, where μ is the Wiener measure, defined $(C[0,1])^{d_1\times d_2}$, and where π_1, π_2 here ranging over all bijective functions, defined on $\{1, 2, \cdots, k_n\}$.

Proof: It is sufficient to prove it in the case where there exists $\varepsilon > 0$, such that $\| \frac{1}{\sqrt{k_n}}s^2(\underline{x}_{k_n}) \| > \varepsilon$, and $\| \frac{1}{\sqrt{k_n}}s^2(\underline{y}_{k_n}) \| > \varepsilon$ are valid for all $n \in \mathbb{N}$. From the new hypotheses above, we conclude that we can reduce this case to the situation of theorem 2.3.5 just by define for each $n \in \mathbb{N}$ these new arrays
$\tilde{x}_{ni} := \left(\frac{1}{\sqrt{k_n}}s^2(\underline{x}_{k_n})\right)^{-\frac{1}{2}}(x_{ni} - \bar{x}_n)$, and $\tilde{y}_{ni} := \left(\frac{1}{\sqrt{k_n}}s^2(\underline{y}_{k_n})\right)^{-\frac{1}{2}}(y_{ni} - \bar{y}_n)$.
Then we have by theorem 2.3.5:

$$\left(\left(\frac{1}{\sqrt{k_n}}s^2(\underline{x}_{k_n})\right)^{-\frac{1}{2}}\sum_{i=1}^{[k_n t]}(\xi_{ni} - \bar{x}_n)(\eta_{ni} - \bar{y}_n)^t\left(\frac{1}{\sqrt{k_n}}s^2(\underline{y}_{k_n})\right)^{-\frac{1}{2}}\right)_{0\leqslant t\leqslant 1} \xrightarrow{\mathcal{D}} W.$$

Consequently, the assertion holds. $\square$

The following theorem is intended to be used later when we shall deal with permutation statistics for the hypothesis of random blocks H_3.

Theorem 2.3.9. For each $n \in \mathbb{N}$, let $x_{n11}, x_{n12}, \cdots, x_{n1k}$; x_{n21}, x_{n22}, $\cdots$, x_{n2k}; $\cdots$; $x_{nn1}, x_{nn2}, \cdots, x_{nnk}$ be a sequence of blocks of d-dimensional arrays of real numbers which are not distinct necessarily, let further (ξ_{n11}, ξ_{n12}, $\cdots$, ξ_{n1k}; ξ_{n21}, ξ_{n22}, $\cdots$, ξ_{n2k}; $\cdots$; $\xi_{nn1}, \xi_{nn2}, \cdots, \xi_{nnk}$) be an array of blocks of random arrays, where the blocks are independent and the random arrays of the i'th block taking values in the set of all permutations of the block $x_{ni1}, x_{ni2}, \cdots, x_{nik}$, where each permutation having the same probability $\frac{1}{k!}$, and that is for $i = 1, \cdots, n$. Let us define X_{nj} by $X_{nj}(t) := \sum_{i=1}^{[nt]} \xi_{nij}$, $X_{nj}(t) := 0$ whenever $0 \leqslant t < 1/n$, $j = 1, \cdots, k$, where $[\cdot]$ stands for the integral value function.
If $\sum_{j=1}^{k} x_{nij} = 0$, $\frac{n}{k} \sum_{j=1}^{k} x_{nij} x_{nij}^t = I_d$, $i = 1, \cdots, n$ then we have $X_{nj} \xrightarrow{\mathcal{D}} W$, $j = 1, \cdots, k$, where W here is a d-dimensional Brownian motion with independent arguments, each argument is a one-dimensional Brownian motion.

Proof: Let us begin with some preliminary computations.
Let $n \in \mathbb{N}$, from the hypotheses we find $E(\xi_{nij}) = \frac{1}{k} \sum\limits_{l=1}^{k} x_{nil} = 0$, and $E\left(\xi_{nij}\xi_{nij}^t\right) = \frac{1}{k} \sum\limits_{l=1}^{k} (x_{nil} x_{nil}^t) = \frac{1}{n} I_d$, $i = 1, \cdots, n$, $j = 1, \cdots, k$. Also, we remark that for each $j = 1, \cdots, k$ the random variables ξ_{n1j}, ξ_{n2j}, $\cdots$, ξ_{nnj} are independent.
$E(\sum_{i=1}^{[nt]} \xi_{nij}) = 0$, $E\left(\left(\sum_{i=1}^{[nt]} \xi_{nij}\right)\left(\sum_{i=1}^{[nt]} \xi_{nij}\right)^t\right) = [nt]/n I_d \longrightarrow t I_d$ for $n \longrightarrow \infty$.
We turn now to the main part of this proof, where we shall implement the same steps of the proof of theorem 2.3.3 for $\rho_m(t) = 0$, $\sigma_m^2(t) = 1$, and $m = 1$, $\cdots$, d. Fix j of $\{1, 2, \cdots, k\}$, we begin verifying the condition (1^o) which states:
If $0 \leqslant t_1 \leqslant \cdots \leqslant t_k < 1$, then

(I) $$\lim_{h \downarrow 0} \limsup_{n \longrightarrow \infty} \frac{1}{h} E\left\{\left| E\left\{ X_{nj}^m(t_k + h) - X_{nj}^m(t_k) \middle| X_{nj}(t_1), \cdots, X_{nj}(t_k)\right\} - \right.\right.$$
$$\left.\left. - h\rho_m(t_k) X_{nj}^m(t_k)\right|\right\} = 0, \text{ for } m = 1, \cdots, d,$$

(II) $$\lim_{h \downarrow 0} \limsup_{n \longrightarrow \infty} \frac{1}{h} E\left\{\left| E\left\{ \left(X_{nj}^m(t_k + h) - X_{nj}^m(t_k)\right)^2 \middle| X_{nj}(t_1), \cdots, X_{nj}(t_k)\right\} - \right.\right.$$
$$\left.\left. - h\sigma_m^2(t_k)\right|\right\} = 0, \text{ for } m = 1, \cdots, d,$$

and

(III) $$\lim_{h\downarrow 0}\limsup_{n\longrightarrow\infty}\frac{1}{h}E\Big\{\Big|E\Big\{\big(X_{nj}^{m_1}(t_k+h)-X_{nj}^{m_1}(t_k)\big)\big(X_{nj}^{m_2}(t_k+h)-X_{nj}^{m_2}(t_k)\big)\big|$$
$$\big|X_{nj}(t_1),\cdots,X_{nj}(t_k)\Big\}\Big|\Big\}=0,\text{ for } m_1\neq m_2.$$

Let us verify (I): Fix $0<t<1$, for reasons of convenience, we put $\bar{m}_1:=[nt]$, and $\bar{m}_2:=[n(t+h)]-[nt]$, where $0<h\leqslant 1-t$. Consequently, for large n we have $1\leqslant\bar{m}_1<\bar{m}_1+\bar{m}_2\leqslant n$.
From the assumptions we know that, $\xi_{n,\bar{m}_1+1,j}$, $\cdots$, $\xi_{n,\bar{m}_1+\bar{m}_2,j}$ are independent of $\xi_{n1j},\cdots,\xi_{n\bar{m}_1j}$. And we have similarly to the beginning of the proof: $E\left\{\sum_{i=\bar{m}_1+1}^{\bar{m}_1+\bar{m}_2}\xi_{nij}\big|\xi_{n1j},\cdots,\xi_{nm_1j}\right\}=0$, which implies $E\left\{X_{nj}^m(t+h)-X_{nj}^m(t)\big|\xi_{n1j},\cdots,\xi_{nm_1j}\right\}=0$, for $m=1,\cdots,d$. Now, for $t=t_k$, we have of course,

$$\sigma(X_{nj}(t_1),\cdots,X_{nj}(t_k))\subseteq\sigma(\xi_{n1j},\cdots,\xi_{nm_1j}).$$

Therefore,
$\frac{1}{h}E\left\{\big|E\left\{X_{nj}^m(t_k+h)-X_{nj}^m(t_k)\big|X_{nj}(t_1),\cdots,X_{nj}(t_k)\right\}-h\cdot 0\cdot X_{nj}^m(t_k)\big|\right\}=0$, for $m=1,\cdots,d$.
Hence, (I) is verified.
Also, for fixed $m=1,\cdots,d$, we have
$E\left\{\left(\sum_{i=\bar{m}_1+1}^{\bar{m}_1+\bar{m}_2}\xi_{nij}^m\right)^2\big|\xi_{n1j},\cdots,\xi_{nm_1j}\right\}=\frac{\bar{m}_2}{n}\underset{n\longrightarrow\infty}{\longrightarrow}h$
Therefore,
$E\left\{\left(X_{nj}^m(t+h)-X_{nj}^m(t)\right)^2\big|\xi_{n1j},\cdots,\xi_{nm_1j}\right\}=\frac{\bar{m}_2}{n}\underset{n\longrightarrow\infty}{\longrightarrow}h.$
And consequently
$$\frac{1}{h}E\left\{\left|E\left\{\left(X_{nj}^m(t_k+h)-X_{nj}^m(t_k)\right)^2\big|X_{nj}(t_1),\cdots,X_{nj}(t_k)\right\}-h\cdot 1\right|\right\}\underset{n\longrightarrow\infty}{\longrightarrow}0.$$

Therefore, also the condition (II) is verified. Also, for fixed $m_1,m_2\in\{1,\cdots,d\}$, and $m_1\neq m_2$, we have
$E\left\{\left(\sum_{i=\bar{m}_1+1}^{\bar{m}_1+\bar{m}_2}\xi_{nij}^{m_1}\right)\left(\sum_{i=\bar{m}_1+1}^{\bar{m}_1+\bar{m}_2}\xi_{nij}^{m_2}\right)\big|\xi_{n1j},\cdots,\xi_{nm_1j}\right\}=0.$
Therefore,
$E\left\{\left(X_{nj}^{m_1}(t+h)-X_{nj}^{m_1}(t)\right)\left(X_{nj}^{m_2}(t+h)-X_{nj}^{m_2}(t)\right)\big|\xi_{n1j},\cdots,\xi_{nm_1j}\right\}=0.$
And consequently
$$\frac{1}{h}E\Big\{\Big|E\Big\{\left(X_{nj}^{m_1}(t+h)-X_{nj}^{m_1}(t)\right)\left(X_{nj}^{m_2}(t+h)-X_{nj}^{m_2}(t)\right)\big|$$
$$\big|X_{nj}(t_1),\cdots,X_{nj}(t_k)\Big\}\Big|\Big\}=0.$$

Therefore, also the condition (III) is verified.
Hence, we have verified the condition (1^o).
Now, we turn to verify the remark following the proof of theorem 2.2.6:
Since the maximum jump $\max_{1\leqslant i\leqslant n}||x_{nij}||$ in X_{nj} tends to zero and $X_{nj}(0) = 0, \forall n \in \mathbb{N}$, the mentioned remark is verified obviously.
Now, we turn to condition(3^o) which states: There is a constant K such that, for all $m = 1, \cdots, d$, and for all $n \in \mathbb{N}$, and also for all $t_1, t, t_2 \in [0, 1]$ satisfying $t_1 \leqslant t \leqslant t_2$, then
$\limsup_{n\longrightarrow\infty} E\left\{\left(X^m_{nj}(t) - X^m_{nj}(t_1)\right)^2 \left(X^m_{nj}(t_2) - X^m_{nj}(t)\right)^2\right\} \leqslant K(t_2 - t_1)^2.$
First, if $t_2 - t_1 \leqslant \frac{1}{n}$, then $E\left\{\left(X^m_{nj}(t) - X^m_{nj}(t_1)\right)^2 \left(X^m_{nj}(t_2) - X^m_{nj}(t)\right)^2\right\} = 0$, and the condition holds.
Now, if $t_2 - t_1 > \frac{1}{n}$, then we find easily that:
$E\left\{\left(X^m_{nj}(t) - X^m_{nj}(t_1)\right)^2 \left(X^m_{nj}(t_2) - X^m_{nj}(t)\right)^2\right\} = \sum\{\xi^m_{ni_1j}\xi^m_{ni_2j}\xi^m_{ni_3j}\xi^m_{ni_4j}\}$, where $[nt_1] < i_1, i_2 \leqslant [nt]$, and $[nt] < i_3, i_4 \leqslant [nt_2]$. Let us put $\bar{m}_1 := [nt] - [nt_1]$ and $\bar{m}_2 := [nt_2] - [nt]$. We have

$$\begin{aligned}
&\sum\{\xi^m_{ni_1j}\xi^m_{ni_2j}\xi^m_{ni_3j}\xi^m_{ni_4j}\} = \\
&= \bar{m}_1\bar{m}_2 E\{(\xi^m_{n1j})^2(\xi^m_{n2j})^2\} + \bar{m}_1\bar{m}_2(\bar{m}_1 + \bar{m}_2 - 2)E\{(\xi^m_{n1j})^2\xi^m_{n2j}\xi^m_{n3j}\} + \\
&\qquad + \bar{m}_1(\bar{m}_1 - 1)\bar{m}_2(\bar{m}_2 - 1)E\{\xi^m_{n1j}\xi^m_{n2j}\xi^m_{n3j}\xi^m_{n4j}\}.
\end{aligned}$$

$= \bar{m}_1\bar{m}_2 E\{(\xi^m_{n1j})^2(\xi^m_{n2j})^2\} + 0$
$= \frac{\bar{m}_1\bar{m}_2}{n^2} \leqslant \frac{1}{4}\frac{(\bar{m}_1+\bar{m}_2)^2}{n^2}.$
$\limsup_{n\longrightarrow\infty} E\left\{\left(X^m_{nj}(t) - X^m_{nj}(t_1)\right)^2 \left(X^m_{nj}(t_2) - X^m_{nj}(t)\right)^2\right\} \leqslant \frac{1}{4}(t_2 - t_1)^2.$
Therefore, the condition (3^o) is verified at least for $K \geqslant \frac{1}{4}$, and consequently the proof is complete. □

Remark 2.3.10. We keep all the hypotheses of theorem 2.3.9, but here we put these new conditions:
For each $n \in \mathbb{N}$, $\sum_{j=1}^k x_{nij}$, and $\sum_{j=1}^k (x_{nij} - \bar{x}_n)(x_{nij} - \bar{x}_n)^t$ are independent of the value of i, where $\bar{x}_n := \frac{1}{k}\sum_{j=1}^k x_{n\cdot j}$, i.e. we can write:

$$\sum_{j=1}^k x_{nij} = \sum_{j=1}^k x_{n\cdot j},$$

$$\sum_{j=1}^k (x_{nij} - \bar{x}_n)(x_{nij} - \bar{x}_n)^t = \sum_{j=1}^k (x_{n\cdot j} - \bar{x}_n)(x_{n\cdot j} - \bar{x}_n)^t.$$

Now, if the sequences $\left(\frac{n}{k}\sum_{j=1}^{k} x_{n\cdot j}\right)_{n\in\mathbb{N}}$, and $\left(\frac{n}{k}\sum_{j=1}^{k}(x_{n\cdot j}-\bar{x}_n)(x_{n\cdot j}-\bar{x}_n)^t\right)_{n\in\mathbb{N}}$ are bounded, and the matrices $s^2(\underline{x}_n)$ are assumed to be positive definite for all $n\in\mathbb{N}$. Then

$$X_{nj} \overset{w}{\sim} \left(\sqrt{\frac{n}{k}}\sqrt{s^2(\underline{x}_n)}W_t + t\frac{n}{k}\sum_{j=1}^{k} x_{n\cdot j}\right)_{0\leqslant t\leqslant 1} \quad (\mathcal{C}_n),\ j=1,2,\cdots,k,$$

where $s^2(\underline{x}_n) := \sum_{j=1}^{k}(x_{n\cdot j}-\bar{x}_n)(x_{n\cdot j}-\bar{x}_n)^t$.
It is important to mention here that the assertion of this remark means by definition:

$$E(\varphi(X_{nj})) - E\left(\varphi\left(\sqrt{\frac{n}{k}}\sqrt{s^2(\underline{x}_n)}W + (\cdot)\frac{n}{k}\sum_{j=1}^{k} x_{n\cdot j}\right)\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0,\ j=1,\cdots,k,$$

or more explicitly,

$$\frac{1}{(k!)^n}\sum_{\pi_1,\pi_2,\cdots,\pi_n}\varphi\left(\sum_{i=1}^{[n(\cdot)]} x_{ni\pi_i(j)}\right) - $$
$$- \int_{(C[0,1])^d}\varphi\left(\sqrt{\tfrac{n}{k}}\sqrt{s^2(\underline{x}_n)}W + (\cdot)\tfrac{n}{k}\sum_{j=1}^{k} x_{n\cdot j}\right)d\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0,\ j=1,\cdots,k.$$

For all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where μ is the Wiener measure, defined $(C[0,1])^d$, and where $\pi_1,\pi_2,\cdots,\pi_n$ here ranging over all bijective functions, defined on $\{1,2,\cdots,k\}$.

Proof: It is sufficient to prove it in the case where there exists $\varepsilon > 0$, such that $\| \frac{n}{k}s^2(\underline{x}_n) \| > \varepsilon$ is valid for all $n\in\mathbb{N}$. We can reduce the case to the situation of the previous corollary by using these new arrays:
$\tilde{x}_{nij} := \left(\frac{n}{k}s^2(\underline{x}_n)\right)^{-\frac{1}{2}}(x_{nij}-\bar{x}_n)$, $n\in\mathbb{N}$, $i=1,\cdots,n$, $j=1,\cdots,k$.
And consequently, we obtain by theorem 2.3.9:

$$\left(\left(\frac{n}{k}s^2(\underline{x}_n)\right)^{-\frac{1}{2}}\sum_{i=1}^{[nt]}(\xi_{nij}-\bar{x}_n)\right)_{0\leqslant t\leqslant 1} \overset{\mathcal{D}}{\longrightarrow} W,\ j=1,\cdots,k.$$

But from the hypotheses the sequences $\left(\frac{n}{k}\sum_{j=1}^{k} x_{n\cdot j}\right)_{n\in\mathbb{N}}$, and $\left(\frac{n}{k}\sum_{j=1}^{k}(x_{n\cdot j}-\bar{x}_n)(x_{n\cdot j}-\bar{x}_n)^t\right)_{n\in\mathbb{N}}$ are bounded.
Hence, the assertion is valid. □

Chapter Three

Conditional Distributions Limit Theorems for Permutation Test Statistics

3.1. The Hypothesis H_0 (randomness):

We want to say here that some of the results, which are written in this section, can be proved directly by using those results in the paper "On the asymptotic theory of permutation statistics [1998]" which is due to H. Strasser and C. Weber, but the proofs which I presented here are in some places either similar to the orginal ones or modified. In fact, the main reason for that is first to combine the case of H_0 with other cases of H_1, H_2, H_3 in one volume, and secondly because the mentioned proofs have been completed and explained entirely here.

Introduction: Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathbb{P}$ be a probability measure defined on $\mathcal{A}$, $\underline{X}_{k_n} = (X_{n1}, X_{n2}, \cdots, X_{nk_n})^t$, $n \in \mathbb{N}$, be a triangular array of random elements with values in a sample space $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. Assume that the random elements of each row (i.e. $X_{n1}, \cdots, X_{nk_n}$, $n \in \mathbb{N}$) are i.i.d. under $\mathbb{P}$. Let the distribution of X_{ni} under $\mathbb{P}$ be denoted by $\mathbb{P} * X_{ni}$ or shortly by P_n. The general form of linear statistics which are considered here is denoted by $T_n\left(\underline{X}_{k_n}\right)$, or shortly by T_n if we can avoid confusion, where $T_n := \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \omega_{ni} f_n\left(X_{ni}, \underline{X}_{k_n}\right)$, and where $\omega_{ni} \in \mathcal{M}_{m\times d}$, $i = 1, \cdots, k_n$, $n \in \mathbb{N}$, these matrices are assumed to be given constants. The functions $f_n : \mathcal{E} \times \mathcal{E}^{k_n} \longrightarrow \mathbb{R}^d$, $(x, \underline{y}_{k_n}) \longmapsto f_n(x, \underline{y}_{k_n})$, $n \in \mathbb{N}$, are measurable and such that they depend on $\underline{y}_{k_n}$ in a permutation symmetric way. This means $f_n\left(x, \pi\left(\underline{y}_{k_n}\right)\right) = f_n(x, \underline{y}_{k_n})$, for all $\pi \in \Pi_{k_n}$, where Π_{k_n} is the set of all bijective functions defined on the set $\{1, 2, \cdots, k_n\}$. Fix $n \in \mathbb{N}$, and let $\lambda \in \mathbb{R}^d$ be a given constant, define $\Omega_\lambda \subseteq \Omega$ by

$$\Omega_\lambda := \left\{ \lambda^t f_n\left(X_{n1}, \underline{X}_{k_n}\right) = \lambda^t f_n\left(X_{n2}, \underline{X}_{k_n}\right) = \cdots = \lambda^t f_n\left(X_{nk_n}, \underline{X}_{k_n}\right) \right\}$$

we assume that $\mathbb{P}(\Omega_\lambda) > 0$ is hold iff $\lambda = 0$. Now, for notational convenience concerning the computations, let $\omega_n \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d})$ be such that

$\omega_{ni} = k_n \int\limits_{\frac{i-1}{k_n}}^{\frac{i}{k_n}} \omega_n(t)dt,\ i = 1, \cdots, k_n,\ n \in \mathbb{N}.$

We mention here that the form of the considered statistic T_n contains the linear rank test statistic $S = \sum_{i=1}^{n} c_i a(R_i)$ which has been discussed deeply in "Theory of rank tests [1967]" which is due to J. Hájek and Z. Šidák.

To achieve the purpose here we need to build a base for our coming limit theorems, for this we shall begin with the following interesting steps which are needed very much to obtain the final results.

Let us here introduce here the known kind of exchangeablity, but we call it here H_0-exchangeablity, also we put some related symbols, which will be needed for the main results of this section.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\underline{\xi}_{k_n} = (\xi_{n1}, \xi_{n2}, \cdots, \xi_{nk_n})^t$ be a triangular array of random elements from $(\Omega, \mathcal{A}, P)$ to $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. And let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence of sub-σ-fields of $\mathcal{A}$, $\mathcal{C}$ be a sub-σ-field of $\mathcal{A}$. Let further $\mathcal{S}^0\left(\underline{\xi}_{k_n}\right) := \left(\underline{\xi}_{k_n}\right)^{-1}(\mathcal{S}_n^0)$, where $\mathcal{S}_n^0$ denotes here the σ-field of all sets B in $\mathcal{B}^{k_n}$ satisfying the condition

$$\underline{x}_{k_n} \in B \Longleftrightarrow \pi\left(\underline{x}_{k_n}\right) \in B,\ \forall \pi \in \Pi_{k_n},$$

where here $\underline{x}_{k_n} := (x_1, \cdots, x_{k_n})$. Let $s^2(\underline{\xi}_{k_n}) := \sum_{i=1}^{k_n} \left(\xi_{ni} - \bar{\xi}_n\right)\left(\xi_{ni} - \bar{\xi}_n\right)^t$, and $\bar{\xi}_n := \frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{ni}$. For each $n \in \mathbb{N}$ the matrix $s^2(\underline{\xi}_{k_n})$ is assumed to be positive definite. These assumptions are hold in this section.

Definition 3.1.1. The random arrays ξ_{n1}, ξ_{n2}, $\cdots$, ξ_{nk_n} are H_0-exchangeable under $P(\cdot|\mathcal{C}_n)$, iff $\forall \pi \in \Pi_{k_n}$ the equality

$$P\left(\underline{\xi}_{k_n} \in B|\mathcal{C}_n\right) = P\left(\pi\left(\underline{\xi}_{k_n}\right) \in B|\mathcal{C}_n\right)\ [P]$$

is valid for all $B \in \mathcal{B}^{k_n}$, and then the array $\underline{\xi}_{k_n}$ is called H_0-exchangeable under $P(\cdot|\mathcal{C}_n)$.

Lemma 3.1.2. Let $f : (\mathcal{E}, \mathcal{B})^{k_n} \longrightarrow (\mathbb{R}, \mathbb{B})$ be a measurable function such that $E(f(\underline{\xi}_{k_n}))$ is well-defined. If $\underline{\xi}_{k_n}$ is H_0-exchangeable under $P(\cdot|\mathcal{C})$, then $\forall A \in \mathcal{S}^0(\underline{\xi}_{k_n}),\ C \in \mathcal{C},\ \pi \in \Pi_n$ the following equality is valid

$$E\left(1_{A\cap C} f\left(\pi\left(\underline{\xi}_{k_n}\right)\right)\right) = E\left(1_{A\cap C} f\left(\underline{\xi}_{k_n}\right)\right).$$

□

Let us denote the σ-field $\sigma(\mathcal{S}^0\left(\underline{\xi}_{k_n}\right), \mathcal{C})$ by $\mathcal{S}^0(\underline{\xi}_{k_n}, \mathcal{C})$. Now, since $\{A \cap C : A \in \mathcal{S}^0\left(\underline{\xi}_{k_n}\right),\ C \in \mathcal{C}\}$ generates the σ-field $\mathcal{S}^0(\underline{\xi}_{k_n}, \mathcal{C})$, we have by lemma 3.1.2 the validity of

$$E\left(1_D f\left(\pi\left(\underline{\xi}_{k_n}\right)\right)\right) = E\left(1_D f\left(\underline{\xi}_{k_n}\right)\right),$$

$\forall D \in \mathcal{S}^0(\underline{\xi}_{k_n}, \mathcal{C}),\ \pi \in \Pi_{k_n}$.
Therefore, the following lemma is just a consequence of lemma 3.1.2.

Lemma 3.1.3. Let $f : (\mathcal{E}, \mathcal{B})^{k_n} \longrightarrow (\mathbb{R}, \mathbb{B})$ be a measurable function such that $E(f(\underline{\xi}_{k_n}))$ is well-defined. If $\underline{\xi}_{k_n}$ is H_0-exchangeable under $P(\cdot|\mathcal{C})$, then

$$E(f(\underline{\xi}_{k_n})|\mathcal{S}^0\left(\underline{\xi}_{k_n}, \mathcal{C}\right)) = \frac{1}{k_n!} \sum_{\pi\in\Pi_{k_n}} f(\pi(\underline{\xi}_{k_n}))\ [P].$$

Let us here introduce the following theorem which is an invariance principle, needed to obtain the limit theorems of this section.

Theorem 3.1.4. Suppose that the triangular array $\underline{\xi}_{k_n}$ is infinitesimal, and the sequences $\left(\sum_{i=1}^{k_n} \xi_{ni}\right)_{n\in\mathbb{N}}$, and $\left(s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$ are stochastically bounded. If $\mathcal{C}_n \supseteq \mathcal{S}^0\left(\underline{\xi}_{k_n}\right)$, $n \in \mathbb{N}$, is a sequence of sub-σ-fields of $\mathcal{A}$, $\tilde{\mathcal{C}}_n := \sigma(C \times (C[0,1])^d : C \in \mathcal{C}_n)$, and if the triangular array $\underline{\xi}_{k_n}$ is H_0-exchangeable under $P(\cdot|\mathcal{C}_n)$ then

$$\tilde{\mathcal{S}}_n \overset{w}{\sim} \tilde{W}\ (\tilde{\mathcal{C}}_n).$$

Where

$$\tilde{\mathcal{S}}_n : \Omega \times (C[0,1])^d \longrightarrow (D[0,1])^d,$$

$$\tilde{\mathcal{S}}_n(\omega_1,\omega_2) := \mathcal{S}_n(\omega_1),$$

$$\left(\mathcal{S}_n(\omega_1)\right)(t) := \sum_{i=1}^{[k_n t]} \xi_{ni}(\omega_1).$$

And

$$\tilde{W} : \Omega \times (C[0,1])^d \longrightarrow (C[0,1])^d \subset (D[0,1])^d,$$

$$\tilde{W}_t(\omega_1,\omega_2) := \left(\sqrt{s^2(\underline{\xi}_{k_n})}\right)(\omega_1) W_t^o(\omega_2) + t\sum_{i=1}^{k_n} \xi_{ni}(\omega_1),$$

$\forall t \in [0,1]$, $\forall (\omega_1,\omega_2) \in \Omega \times (C[0,1])^d$, where $W^o := (W_t^o)_{0\leqslant t\leqslant 1}$ is a d-dimensional Brownian bridge.

Proof: We first prove it in the case where $\sup_n \parallel s^2(\underline{\xi}_{k_n}) \parallel < +\infty$ $[P]$, and $\sup_n \parallel \sum_{i=1}^{k_n} \xi_{ni} \parallel < +\infty$ $[P]$.
By lemma 1.2.2 the assertion is equivalent to the validity of the limit

$$\int\limits_{C_n \times C[0,1]^d} \varphi(\tilde{\mathcal{S}}_n) dP \otimes \mu - \int\limits_{C_n \times C[0,1]^d} \varphi(\tilde{W}) dP \otimes \mu \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

for all sequences $(C_n)_{n\in\mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where $C_n \in \mathcal{C}_n$, and where μ is the the Wiener measure defined on $(C[0,1])^d$.
Therefore, let $(C_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of the measurable sets $C_n \in \mathcal{C}_n$, $n \in \mathbb{N}$, we want to prove the validity of the previous limit above, which is equivalent to

$$\int\limits_{C_n} \varphi(\mathcal{S}_n) dP - \int\limits_{C_n} \int\limits_{(C[0,1])^d} \varphi(\tilde{W}) dP d\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0$$

$$\Longleftrightarrow \int\limits_{C_n} E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right) dP - \int\limits_{C_n} E\left(\int\limits_{(C[0,1])^d} \varphi(\tilde{W}) d\mu \middle| \mathcal{C}_n \right) dP \underset{n\longrightarrow\infty}{\longrightarrow} 0$$

Since the sequence $(C_n)_{n\in\mathbb{N}}$ is arbitrary, the previous assertion is equivalent to

$$E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right) - E\left(\int\limits_{(C[0,1])^d} \varphi(\tilde{W}) d\mu \middle| \mathcal{C}_n \right) \xrightarrow{P} 0$$

$$\Longleftrightarrow E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right) - \int\limits_{(C[0,1])^d} \varphi(\tilde{W})d\mu \overset{P}{\longrightarrow} 0$$

$$\Longleftrightarrow \frac{1}{k_n!}\sum_{\pi}\varphi\left(\sum_{i=1}^{[k_n(\cdot)]}\xi_{n\pi(i)}\right) - \int\limits_{(C[0,1])^d} \varphi(\tilde{W})d\mu \overset{P}{\longrightarrow} 0,$$

where π here ranging over all bijective functions, defined on $\{1, 2, \cdots, k_n\}$. It is sufficient to prove that each subsequence $\{n'\}$ contains another sub-subsequence $\{n''\}$ such that

$$\frac{1}{k_{n''}!}\sum_{\pi}\varphi\left(\sum_{i=1}^{[k_{n''}(\cdot)]}\xi_{n''\pi(i)}\right) - \int\limits_{(C[0,1])^d} \varphi(\tilde{W})d\mu \longrightarrow 0\ [P].$$

From the hypotheses, each sub-sequence $\{n'\}$ of $\{n\}$ contains another sub-subsequence $\{n''\}$ such that $\max\limits_{1\leqslant i\leqslant k_{n''}} \parallel \xi_{n''i} \parallel \longrightarrow 0\ [P]$. Therefore, without any loss of generality we can assume that $\max\limits_{1\leqslant i\leqslant k_n} \parallel \xi_{ni} \parallel \longrightarrow 0\ [P]$. Hence, for all ω_1 of those fulfill the hypotheses, we want now to prove

$$\frac{1}{k_n!}\sum_{\pi}\varphi\left(\sum_{i=1}^{[k_n(\cdot)]}\xi_{n\pi(i)}(\omega_1)\right) - \\ - \int\limits_{(C[0,1])^d} \varphi\left(\left(\sqrt{s^2(\underline{\xi}_{k_n})}\right)(\omega_1)W^o + (\cdot)\sum_{i=1}^{k_n}\xi_{ni}(\omega_1)\right)d\mu \longrightarrow 0.$$

But this is valid indeed from remark 2.3.2.
Now, we turn to prove it in the general case.
Let us define $C_n^M := \left\{ \parallel \sum\limits_{i=1}^{k_n}\xi_{ni} \parallel\leqslant M,\ \parallel s^2(\underline{\xi}_{k_n}) \parallel\leqslant M \right\}$, and also for each $n \in \mathbb{N}$, and $i = 1, \cdots, k_n$, we define the new random arrays $\tilde{\xi}_{ni} := 1_{C_n^M}\cdot\xi_{ni}$. It is clear that the new random arrays satisfy the first case of the proof above. Consequently, the limit

$$\int\limits_{C_n\times(C[0,1])^d} \varphi(1_{C_n^M}\cdot\tilde{\mathcal{S}}_n)dP\otimes\mu - \int\limits_{C_n\times(C[0,1])^d} \varphi(1_{C_n^M}\cdot\tilde{W})dP\otimes\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

is valid for all sequences $(C_n)_{n\in\mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where $C_n \in \mathcal{C}_n$. We can rewrite the previous limit as the following

$$\int\limits_{C_n\times(C[0,1])^d} 1_{C_n^M}\cdot\varphi(\tilde{\mathcal{S}}_n)dP\otimes\mu - \int\limits_{C_n\times(C[0,1])^d} 1_{C_n^M}\cdot\varphi(\tilde{W})dP\otimes\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

Also, we can enlarge M to make the following inequality valid $P(C_n^M) \geqslant 1-\varepsilon$ for any given $\varepsilon > 0$, and for all $n \in \mathbb{N}$.
Therefore, we obtain the validity of the limit

$$\int\limits_{C_n\times(C[0,1])^d} \varphi(\tilde{\mathcal{S}}_n)dP\otimes\mu - \int\limits_{C_n\times(C[0,1])^d} \varphi(\tilde{W})dP\otimes\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

for all sequences $(C_n)_{n\in\mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where $C_n \in \mathcal{C}_n$. Hence, the proof is complete. □

Corollary 3.1.5. In the same situation of theorem 3.1.4, we have the validity of

$$\tilde{\mathcal{S}}_n \overset{w}{\sim} \tilde{W},$$

or more explicitly

$$\left(\sum_{i=1}^{[k_nt]} \xi_{ni}\right)_{0\leqslant t\leqslant 1} \overset{w}{\sim} \left(\left(\sqrt{s^2(\underline{\xi}_{k_n})}\right) W_t^o + t\sum_{i=1}^{k_n} \xi_{ni}\right)_{0\leqslant t\leqslant 1}.$$

In the same situation of theorem 3.1.4, we want to discuss the conditional asymptotic normality of a given linear combination of the random arrays $S_n(t_1)$, $S_n(t_2)$,$\cdots$, $S_n(t_k)$, where t_1, t_2,$\cdots$, t_k belong to $[0,1]$. The following remark is for this purpose.

Remark 3.1.6. Let X_n be defined by $\begin{pmatrix} S_n(t_1) \\ \vdots \\ S_n(t_k) \end{pmatrix}$, where t_1, t_2,$\cdots$, t_k belong to $[0,1]$, and let a_1, a_2,$\cdots$, a_k be given constant matrices of order $m \times d$, then we have
$P * \sum\limits_{i=1}^{k} a_i \cdot S_n(t_i) \sim \mathcal{N}(\mu_n, \sigma_n^2)$ $(\mathcal{C}_n)$, where $\forall n \in \mathbb{N}$

$$\mu_n := \sum_{j=1}^{k} t_j a_j \cdot \left(\sum_{i=1}^{k_n} \xi_{ni}\right)$$

$$\sigma_n^2 := (a_1, \cdots, a_k) \cdot \left(\min(t_i,t_j)(1-\max(t_i,t_j)s^2\left(\underline{\xi}_{k_n}\right)\right)_{1\leqslant i,j\leqslant k} \cdot \begin{pmatrix} a_1^t \\ \vdots \\ a_k^t \end{pmatrix}$$

Proof: From Theorem 3.1.4 we conclude the following fact

$$P * \begin{pmatrix} S_n(t_1) \\ \vdots \\ S_n(t_k) \end{pmatrix} \sim \mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right) \ (\mathcal{C}_n),$$

where here $\forall n \in \mathbb{N}$

$$\tilde{\mu}_n := \begin{pmatrix} t_1 \sum_{i=1}^{k_n} \xi_{ni} \\ \vdots \\ t_k \sum_{i=1}^{k_n} \xi_{ni} \end{pmatrix},$$

$$\tilde{\sigma}_n^2 := \left(\min(t_i, t_j)(1 - \max(t_i, t_j)) s^2(\underline{\xi}_{k_n}) \right)_{1 \leqslant i,j \leqslant k}.$$

Which means by definition the validity of the following limit $E_P\left(f\left(S_n(t_1), S_n(t_2), \cdots, S_n(t_k)\right) | \mathcal{C}_n\right) - \int f d\mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right) \xrightarrow{P} 0$, for all bounded uniformly continuous functions $f : (\mathbb{R}^d)^k \longrightarrow \mathbb{R}$. Now, let $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a bounded uniformly continuous function, and let $a_1, a_2, \cdots, a_k \in \mathcal{M}_{m \times d}$ be given constant matrices, then the function $h : (\mathbb{R}^d)^k \longrightarrow \mathbb{R}$, defined by $h(x) := g((a_1, a_2, \cdots, a_k) \cdot x)$, $\forall x \in (\mathbb{R}^d)^k$ is also bounded uniformly continuous function. Thus, we have for this h the validity of the limit $E_P\left(h\left(S_n(t_1), S_n(t_2), \cdots, S_n(t_k)\right) | \mathcal{C}_n\right) - \int h d\mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right) \xrightarrow{P} 0$, which can be rewritten as $E_P\left(g\left((a_1, a_2, \cdots, a_k) \cdot \begin{pmatrix} S_n(t_1) \\ \vdots \\ S_n(t_k) \end{pmatrix} \right) \middle| \mathcal{C}_n \right) - \int g d\mathcal{N}\left(\mu_n, \sigma_n^2\right) \xrightarrow{P} 0$, which is also equivalent to

$$E_P\left(g\left(\sum_{j=1}^{k} a_j \cdot S_n(t_j) \right) | \mathcal{C}_n \right) - \int g d\mathcal{N}\left(\mu_n, \sigma_n^2\right) \xrightarrow{P} 0,$$

where here

$$\mu_n := (a_1, a_2, \cdots, a_k) \cdot \begin{pmatrix} t_1 \sum_{i=1}^{k_n} \xi_{ni} \\ \vdots \\ t_k \sum_{i=1}^{k_n} \xi_{ni} \end{pmatrix} = \sum_{j=1}^{k} t_j a_j \cdot \left(\sum_{i=1}^{k_n} \xi_{ni} \right),$$

and

$$\sigma_n^2 := (a_1, \cdots, a_k) \cdot \left(\min(t_i, t_j)(1 - \max(t_i, t_j)s^2\left(\underline{\xi}_{k_n}\right)\right)_{1\leqslant i,j\leqslant k} \cdot \begin{pmatrix} a_1^t \\ \vdots \\ a_k^t \end{pmatrix}.$$

Therefore, the proof is complete. □

Corollary 3.1.7. In remark 3.1.6 if $0 < t_1 < t_2 < \cdots < t_k < 1$, then we can rewrite σ_n^2 as the following

$$\sigma_n^2 = \sum_{j=1}^{k}\left(\sum_{i=1}^{j} a_i s^2(\underline{\xi}_{k_n})a_j^t t_i(1-t_j) + \sum_{i=1}^{k-j} a_{j+i}s^2(\underline{\xi}_{k_n})a_j^t t_j(1-t_{j+i})\right).$$

Also, if $t = \frac{1}{k}$, $a_j = \alpha_j - \alpha_{j+1}$, and $a_k = \alpha_k$, where α_1, α_2,$\cdots$, α_k, are given constant matrices of order $m \times d$, and if $t_j = jt$, for $j = 1, 2, \cdots, k$, then we have easily
$\mu_n = t\left(\sum\limits_{j=1}^{k}\alpha_j\right)\cdot\left(\sum\limits_{i=1}^{k_n}\xi_{ni}\right)$, and
$\sigma_n^2 = \left(\sum\limits_{j=1}^{k}\alpha_j s^2\left(\underline{\xi}_{k_n}\right)\alpha_j^t\right)t - \left(\sum\limits_{j=1}^{k}\alpha_j\right)s^2\left(\underline{\xi}_{k_n}\right)\left(\sum\limits_{j=1}^{k}\alpha_j\right)^t t^2$. These results are involved in the proof of theorem 3.1.9. □

Throughout the proofs of the coming limit theorems in this section we need to represent some Partial sums as path integrals. Let ξ_{n1}, ξ_{n2},$\cdots$,ξ_{nk_n} be random arrays from $(\Omega, \mathcal{A}, P)$ to $(\mathbb{R}^d, \mathbb{B}^d)$, $S_n(t) := \sum\limits_{i=1}^{[k_n t]}\xi_{ni}$, and let ω_{ni}, $i = 1, 2, \cdots, k_n$, $n \in \mathbb{N}$, be constant matrices of order $m \times d$. One can write $\sum\limits_{i=1}^{k_n}\omega_{ni}\xi_{ni} = \int_0^1 \omega_n(t)dS'(t)$, where $S'_n(t)$ is defined by $S'_n(t) = S_n\left(\frac{i-1}{k_n}\right) + k_n\left(t - \frac{i-1}{k_n}\right)\left(S_n\left(\frac{i}{k_n}\right) - S_n\left(\frac{i-1}{k_n}\right)\right)$, where $i := [k_n t] + 1$ in this formula, and $\omega_n \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d})$ is defined such that $\omega_{ni} = k_n \int\limits_{\frac{i-1}{k_n}}^{\frac{i}{k_n}} \omega_n(t)dt$, for $i = 1, \cdots, k_n$. Also, we need to see the behavior of $\parallel \sum\limits_{i=1}^{k_n}\omega_{ni}\xi_{ni} \parallel$ when $\parallel \omega_n \parallel_2$ tends to zero with n tends to infinity, the next lemma gives an answer to that question.

Lemma 3.1.8. If the random arrays ξ_{n1}, $\xi_{n2}, \cdots, \xi_{nk_n}$ are H_0-exchangeable under $P(\cdot|\mathcal{C}_n)$, and the sequences $\left(\sum_{i=1}^{k_n} \xi_{ni}\right)_{n\in\mathbb{N}}$, and $\left(s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$ are stochastically bounded, then

$$\| \sum_{i=1}^{k_n} \omega_{ni}\xi_{ni} \| \xrightarrow{P} 0, \text{ if } \| \omega_n \|_2 \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

Proof: Since the sequence $\left(\sum_{i=1}^{k_n} \xi_{ni}\right)_{n\in\mathbb{N}}$ is stochastically bounded, it is sufficient to prove

$$\| \sum_{i=1}^{k_n} \omega_{ni}\left(\xi_{ni} - \bar{\xi}_n\right) \| \xrightarrow{P} 0, \text{ if } \| \omega_n \|_2 \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

where as we know $\bar{\xi}_n := \frac{1}{k_n}\sum_{i=1}^{k_n} \xi_{ni}$.
Since $\omega \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d})$, and from Lemma 3.1.3, we have

$$E\left(\left(\sum_{i=1}^{k_n} \omega_{ni}\left(\xi_{ni} - \bar{\xi}_n\right)\right)\left(\sum_{i=1}^{k_n} \omega_{ni}\left(\xi_{ni} - \bar{\xi}_n\right)\right)^t \middle| \mathcal{S}^0\left(\underline{\xi}_{k_n}, \mathcal{C}_n\right)\right) =$$

$$= \frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni}s^2(\underline{\xi}_{k_n})\omega_{ni}^t - \frac{1}{(k_n)^2}\sum_{i=1}^{k_n} \omega_{ni}s^2(\underline{\xi}_{k_n})\sum_{i=1}^{k_n} \omega_{ni}^t + o_P(1).$$

Now, since

$$\| \frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni} \| = \| \int_0^1 \omega_n(t)dt \| \leqslant \| \omega_n \|_2,$$

and

$$\frac{1}{k_n}\sum_{i=1}^{k_n} \| \omega_{ni} \|^2 \leqslant \int_0^1 \| \omega_n \|^2 \, dt = \| \omega_n \|_2^2,$$

and since the sequence $\left(s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$ is stochastically bounded, we obtain

$$E\left(\left(\sum_{i=1}^{k_n} \omega_{ni}\left(\xi_{ni} - \bar{\xi}_n\right)\right)\left(\sum_{i=1}^{k_n} \omega_{ni}\left(\xi_{ni} - \bar{\xi}_n\right)\right)^t \middle| \mathcal{S}^0\left(\underline{\xi}_{k_n}, \mathcal{C}_n\right)\right) \xrightarrow{P} 0.$$

Therefore,

$$E\left(\| \sum_{i=1}^{k_n} \omega_{ni}\left(\xi_{ni} - \bar{\xi}_n\right) \|^2 \middle| \mathcal{S}^0\left(\underline{\xi}_{k_n}, \mathcal{C}_n\right)\right) \xrightarrow{P} 0,$$

and from theorem 1.1.3 we conclude that the assertion is valid indeed. $\square$

In the next theorem we discuss the asymptotic normality of the sum $\sum_{i=1}^{k_n} \omega_{ni}\xi_{ni}$.

Theorem 3.1.9. Suppose that the triangular array $\underline{\xi}_{k_n}$ is infinitesimal and that the sequences $\left(\sum_{i=1}^{k_n} \xi_{ni}\right)_{n\in\mathbb{N}}$, and $\left(s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$ are stochastically bounded, and further that sequence $(\omega_n)_{n\in\mathbb{N}}$ is a relatively compact sequence in $\mathcal{L}^2([0,1],\mathcal{M}_{m\times d})$. We call it a sequence of weight functions and assume that these weights are constants. If $\mathcal{C}_n \supseteq \mathcal{S}^0\left(\underline{\xi}_{k_n}\right)$, $n \in \mathbb{N}$, is a sequence of sub-σ-fields of $\mathcal{A}$, and if the rows of the triangular array are H_0-exchangeable under $P(\cdot|\mathcal{C}_n)$, then we have

$$P * \sum_{i=1}^{k_n} \omega_{ni}\xi_{ni} \sim \mathcal{N}(\mu_n, \sigma_n^2)\ (\mathcal{C}_n),$$

where

$$\mu_n := \int_0^1 \omega_n(t)dt \cdot \sum_{i=1}^{k_n} \xi_{ni},$$

and

$$\sigma_n^2 := \int_0^1 \omega_n(t)s^2(\underline{\xi}_{k_n})\omega_n^t(t)dt - \int_0^1 \omega_n(t)dt \cdot s^2(\underline{\xi}_{k_n}) \cdot \int_0^1 \omega_n^t(t)dt.$$

Proof: Since, $(\omega_n(t))_{n\in\mathbb{N}}$ is relatively compact, so every subsequence of $(\omega_n(t))_{n\in\mathbb{N}}$ contains another convergent sub-subsequence, and consequently it is sufficient to prove the assertion in the case when the sequence $(\omega_n(t))_{n\in\mathbb{N}}$ is convergent. And from lemma 3.1.8 we conclude that it is sufficient to prove it in the case $\omega_n(t) = \omega(t)$, $\forall n \in \mathbb{N}$. Moreover, remarking that any $\omega \in \mathcal{L}^2([0,1],\mathcal{M}_{m\times d})$ can assumed as a limit of some sequence of step-functions belong to $M_{m\times d}$, where $M_{m\times d} := \left\{\omega : \omega(t) = \sum_{i=1}^{k} x_i \cdot 1_{[\frac{i-1}{k},\frac{i}{k})}(t), x_i \in \mathcal{M}_{m\times d},\ i = 1,\cdots,k,\ k \in \mathbb{N}\right\}$. By applying lemma 3.1.8 we find that it is sufficient to prove it when $\omega \in M_{m\times d}$. Now, we note that $\sum_{i=1}^{k_n} \omega_{ni}\xi_{ni} = \int_0^1 \omega(t)dS_n'(t) = \sum_{i=1}^{k} x_i \left(S_n'\left(\frac{i}{k}\right) - S_n'\left(\frac{i-1}{k}\right)\right) = \sum_{i=1}^{k} a_i S_n'\left(\frac{i}{k}\right)$, where

$a_i := x_i - x_{i+1}$, $i = 1, 2, \cdots, k-1$, $a_k = x_k$. Since the triangular array $\underline{\xi}_{k_n}$ is infintesimal, and from remark 3.1.6, and corollary 3.1.7, we find that the assertion is valid indeed. $\square$

Remark 3.1.10. In theorem 3.1.9, σ_n^2 can be rewritten as
$\sigma_n^2 = \frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni} s^2(\underline{\xi}_{k_n})\omega_{ni}^t - \frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni} s^2(\underline{\xi}_{k_n})\frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni}^t$, $n \in \mathbb{N}$. $\square$

We want at this position to introduce some important properties of the concepts of the H_0-exchangeablity under $P(\cdot|\mathcal{C})$, where $\mathcal{C}$ is a given sub-σ-field of $\mathcal{A}$, also we want to discuss some properties of the σ-field $\mathcal{S}^0\left(\underline{\xi}_k\right)$, $k \in \mathbb{N}$. For this, we shall introduce the definition of the H_0-exchangeablity under P, also some interesting results.

Definition 3.1.11. Under the same hypotheses of definition 3.1.1, we call the triangular array $\underline{\xi}_{k_n}$ H_0-exchangeable under P iff this triangular array is H_0-exchangeable under $P(\cdot|\mathcal{C})$, for $\mathcal{C} = \{\emptyset, \Omega\}$.

– It is clear from lemma 3.1.3 that the H_0-exchangeablity under $P(\cdot|\mathcal{C})$, for all $\mathcal{C} \subseteq \mathcal{S}^0\left(\underline{\xi}_{k_n}\right)$ is equivalent to the H_0-exchangeablity under P.

Lemma 3.1.12. Let $X_1, X_2, \cdots, X_k$ be random elements from $(\Omega, \mathcal{A}, P)$ to $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. Let further, $f : \mathcal{E} \times \mathcal{E}^k \longrightarrow \mathbb{R}^m$ be a measurable function, which is permutation symmetric in its second variable, and let $\xi_i := f(X_i, \underline{X}_k)$, $i = 1, \cdots, k$, then $\mathcal{S}^0\left(\underline{\xi}_k\right) \subseteq \mathcal{S}^0(\underline{X}_k)$.

Proof: It is sufficient to prove that the following implication

$$g\left(\underline{\xi}_k\right) \text{ is } \mathcal{S}^0\left(\underline{\xi}_k\right)\text{-measurable} \Longrightarrow g\left(\underline{\xi}_k\right) \text{ is } \mathcal{S}^0(\underline{X}_k)\text{-measurable}$$

is valid for all measurable $g : (\mathbb{R}^m)^k \longrightarrow \mathbb{R}$. For this, let $h : \mathcal{E}^k \longrightarrow (\mathbb{R}^m)^k$ be defined by $h(\underline{x}_k) := (f(x_1, \underline{x}_k), \cdots, f(x_k, \underline{x}_k))$, where $\underline{x}_k = (x_1, \cdots, x_k) \in \mathcal{E}^k$. Let $g\left(\underline{\xi}_k\right)$ be $\mathcal{S}^0\left(\underline{\xi}_k\right)$-measurable, this implies $g\left(\pi\left(\underline{\xi}_k\right)\right) = g\left(\underline{\xi}_k\right)$ for all $\pi \in \Pi_k$. Let us define $g_0 : \mathcal{E}^k \longrightarrow \mathbb{R}$ by $g_0(\underline{x}_k) := g(h(\underline{x}_k))$.
Hence, $g_0(\underline{X}_k) = g(h(\underline{X}_k)) = g\left(\underline{\xi}_k\right)$, and $g_0(\pi(\underline{X}_k)) = g(h(\pi(\underline{X}_k))) =$

$g\left(\pi\left(\underline{\xi}_k\right)\right) = g\left(\underline{\xi}_k\right) = g_0\left(\underline{X}_k\right)$.
Hence, $g_0\left(\pi\left(\underline{X}_k\right)\right) = g_0\left(\underline{X}_k\right)$, but this means that $g_0\left(\underline{X}_k\right)$ is $\mathcal{S}^0\left(\underline{X}_k\right)$-measurable, and also $g\left(\underline{\xi}_k\right)$ is $\mathcal{S}^0\left(\underline{X}_k\right)$-measurable. Consequently, the assertion is valid indeed. □

Lemma 3.1.13. Let $f : \mathcal{E} \times \mathcal{E}^k \longrightarrow \mathbb{R}^m$ be a measurable function which is permutation symmetric in its second variable. If the random elements $X_1, \cdots, X_k$ are H_0-exchangeable under $P(\cdot|\mathcal{C})$, then $f\left(X_1, \underline{X}_k\right), \cdots, f\left(X_k, \underline{X}_k\right)$, are also H_0-exchangeable under $P(\cdot|\mathcal{C})$.

Proof: It is sufficient to prove that for any measurable function $g : \left(\mathbb{R}^m, \mathbb{B}^m\right)^k \longrightarrow (\mathbb{R}, \mathbb{B})$ satisfying that the quantity
$E\left(g\left(f\left(X_1, \underline{X}_k\right), \cdots, f\left(X_k, \underline{X}_k\right)\right)\right)$ is well-defined, the equality

$$E\left(g\left(\pi\left(f\left(X_1, \underline{X}_k\right), \cdots, f\left(X_k, \underline{X}_k\right)\right)\right)|\mathcal{C}\right) = \\ = E\left(g\left(f\left(X_1, \underline{X}_k\right), \cdots, f\left(X_k, \underline{X}_k\right)\right)|\mathcal{C}\right) \ [P]$$

is valid. For this, let us define $h : \mathcal{E}^k \longrightarrow \left(\mathbb{R}^m\right)^k$ by
$h(\underline{x}_k) = \left(f\left(x_1, \underline{x}_k\right), \cdots, f\left(x_k, \underline{x}_k\right)\right)^t$, where $\underline{x}_k = (x_1, \cdots, x_k) \in \mathcal{E}^k$. And we can write
$h(\underline{X}_k) = \left(f\left(X_1, \underline{X}_k\right), \cdots, f\left(X_k, \underline{X}_k\right)\right)^t$. Consequently, it remains to be proved the validity of the equality

$$E\left(g\left(\pi\left(h\left(\underline{X}_k\right)\right)\right)|\mathcal{C}\right) = E\left(g\left(h\left(\underline{X}_k\right)\right)|\mathcal{C}\right) \ [P].$$

But easily, we see that the left side is equal to $E\left(g\left(h\left(\pi\left(\underline{X}_k\right)\right)\right)|\mathcal{C}\right)$ almost sure with respect to P. Since $X_1, \cdots, X_k$ are H_0-exchangeable under $P(\cdot|\mathcal{C})$, the assertion follows. □

Lemma 3.1.14. Let $\xi_1, \cdots, \xi_k : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^m, \mathbb{B}^m)$ be random arrays, and $\mathcal{C} \subseteq \mathcal{A}$ be a sub-σ-field. Let $\underline{x}_k = (x_1, \cdots, x_k) \in (\mathbb{R}^m)^k$, $\bar{x} = \frac{1}{k}\sum_{i=1}^{k} x_i$, and $V\left(\underline{x}_k\right) = \sum_{i=1}^{k}(x_i - \bar{x})(x_i - \bar{x})^t$.
Assume that $V\left(\underline{x}_k\right)$ is positive definite, and let $g : \mathbb{R}^m \times (\mathbb{R}^m)^k \longrightarrow \mathbb{R}^m$ be defined by $g(y, \underline{x}_k) = \left(V\left(\underline{x}_k\right)\right)^{-\frac{1}{2}}(y - \bar{x})$, and $\eta_i = g\left(\xi_i, \underline{\xi}_k\right)$, $i = 1, \cdots, k$. If $\xi_1, \cdots, \xi_k$ are H_0-exchangeable under $P(\cdot|\mathcal{C})$, and if $\bar{\xi}$ and $V(\underline{\xi}_k)$ are $\mathcal{C}$-measurable, then $\mathcal{S}^0\left(\underline{\xi}_k, \mathcal{C}\right) = \mathcal{S}^0\left(\underline{\eta}_k, \mathcal{C}\right)$.

Proof: First we see that $\mathcal{S}^0\left(\underline{\eta}_k, \mathcal{C}\right) \subseteq \mathcal{S}^0\left(\underline{\xi}_k, \mathcal{C}\right)$ follows from lemma 3.1.12 because g is permutation symmetric in its second variable. It remains

to show that $\mathcal{S}^0\left(\underline{\xi}_k,\mathcal{C}\right) \subseteq \mathcal{S}^0\left(\underline{\eta}_k,\mathcal{C}\right)$. And for this it is sufficient to prove that $\mathcal{S}^0\left(\underline{\xi}_k\right) \subseteq \mathcal{S}^0\left(\underline{\eta}_k,\mathcal{C}\right)$. Let $f:(\mathbb{R}^m)^k \longrightarrow \mathbb{R}$ be any measurable function such that $f\left(\underline{\xi}_k\right)$ is $\mathcal{S}^0\left(\underline{\xi}_k\right)$-measurable. This means $f\left(\pi\left(\underline{\xi}_k\right)\right) = f\left(\underline{\xi}_k\right)$ for all $\pi \in \Pi_k$.
To complete this proof we have to prove that $f\left(\underline{\xi}_k\right)$ is $\mathcal{S}^0\left(\underline{\eta}_k,\mathcal{C}\right)$-measurable. For this, let $z=(z_1,\cdots,z_k) \in (\mathbb{R}^m)^k$, $a \in \mathbb{R}^m$, $B \in \mathcal{M}_{m\times m}$, and B is positive definite and symmetric, h be defined by $h(z,a,B) = (B^{\frac{1}{2}}z_1 + a,\cdots,B^{\frac{1}{2}}z_k+a)^t$. It is easy to see that $h(\pi(z),a,B) = \pi(h(z,a,B))$, $\underline{\xi}_k = h(\underline{\eta}_k,\bar{\xi},V(\underline{\xi}_k))$, and $f\left(\underline{\xi}_k\right) = f\left(h\left(\underline{\eta}_k,\bar{\xi},V(\underline{\xi}_k)\right)\right)$ but $\bar{\xi}$ and $V\left(\underline{\xi}_k\right)$ are $\mathcal{S}^0\left(\underline{\eta}_k,\mathcal{C}\right)$-measurable because $\mathcal{C} \subseteq \mathcal{S}^0\left(\underline{\eta}_k,\mathcal{C}\right)$, and they are $\mathcal{C}$-measurable. But also, $f\left(h\left(\pi\left(\underline{\eta}_k\right),\bar{\xi},V(\underline{\xi}_k)\right)\right) = f\left(h\left(\underline{\eta}_k,\bar{\xi},V(\underline{\xi}_k)\right)\right)$ for all $\pi \in \Pi_k$.
Thus, $f\left(\underline{\xi}_k\right) = f\left(h\left(\underline{\eta}_k,\bar{\xi},V(\underline{\xi}_k)\right)\right)$ is $\mathcal{S}^0\left(\underline{\eta}_k,\mathcal{C}\right)$-measurable. □

In the introduction of this section we have defined the statistic $T_n = \frac{1}{\sqrt{k_n}}\sum_{i=1}^{k_n}\omega_{ni}f_n\left(X_{ni},\underline{X}_{k_n}\right)$, we want here to make some preliminary computations to prepare for the next theorem which will discuss the conditional asymptotic normality of the sequence $(T_n)_{n\in\mathbb{N}}$. For this, let us first prove that $E\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)$, and $Var\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)$, are independent of i. One can easily see

$$E\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) = \frac{1}{k_n}\sum_{i=1}^{k_n} f_n\left(X_{ni},\underline{X}_{k_n}\right) \ [\mathbb{P}],$$

$$\begin{aligned}&Var\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) = \\ &= \tfrac{1}{k_n}\sum_{i=1}^{k_n}\left(f_n\left(X_{ni},\underline{X}_{k_n}\right) - E\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)\cdot \\ &\qquad \cdot\left(f_n\left(X_{ni},\underline{X}_{k_n}\right) - E\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)^t \ [\mathbb{P}].\end{aligned}$$

Therefore, for notational convenience we denote $E\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)$ by $E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)$, i.e.

$$E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) := E\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right),$$

and $Var\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)$ by $Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)$, i.e.

$$Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) := Var\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right).$$

Let us now give explicit expressions for $E\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)$, and $Var\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)$.

$$E\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) = \frac{1}{\sqrt{k_n}}\sum_{i=1}^{k_n}\omega_{ni}E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) \ [\mathbb{P}],$$

$$Var\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) = Var\left(\tfrac{1}{\sqrt{k_n}}\sum_{i=1}^{k_n}\omega_{ni}f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) =$$
$$= \tfrac{1}{k_n}\sum_{i=1}^{k_n}\omega_{ni}Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\omega_{ni}^t +$$
$$+\tfrac{1}{k_n}\sum_{1\leqslant i\neq j\leqslant k_n}\omega_{ni}E\Big(\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)\cdot$$
$$\cdot\left(f_n\left(X_{nj},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)^t|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\Big)\omega_{nj}^t \ [\mathbb{P}].$$

But on the other hand when $i \neq j$ we have

$$E\Big(\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)\cdot$$
$$\cdot\left(f_n\left(X_{nj},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)^t|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\Big) =$$
$$= \tfrac{1}{k_n(k_n-1)}\sum_{1\leqslant r\neq s\leqslant k_n}\left(f_n\left(X_{nr},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)\cdot$$
$$\cdot\left(f_n\left(X_{ns},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)^t \ [\mathbb{P}]$$
$$= \tfrac{1}{k_n(k_n-1)}\sum_{1\leqslant r,s\leqslant k_n}\left(f_n\left(X_{nr},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)\cdot$$
$$\cdot\left(f_n\left(X_{ns},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)^t -$$
$$-\tfrac{1}{k_n(k_n-1)}\sum_{1\leqslant r\leqslant k_n}\left(f_n\left(X_{nr},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)\cdot$$
$$\cdot\left(f_n\left(X_{nr},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)^t \ [P]$$
$$= 0 - \tfrac{1}{k_n-1}Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) \ [P].$$

Hence,

$$Var\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) = \tfrac{1}{k_n-1}\sum_{i=1}^{k_n}\omega_{ni}Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\omega_{ni}^t -$$
$$-\tfrac{1}{k_n-1}\sum_{i=1}^{k_n}\omega_{ni}Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\tfrac{1}{k_n}\sum_{j=1}^{k_n}\omega_{nj}^t \ [P].$$

– For the next theorem we assume first the same situation in the introduction of this section.

Theorem 3.1.15. Assume that the triangular array

$$\frac{1}{\sqrt{k_n}}\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right),\ 1\leqslant i\leqslant k_n,\ n\in\mathbb{N},$$

is infinitesimal and the sequence $\left(Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)_{n\in\mathbb{N}}$ is stochastically bounded. Assume further that the sequence $(\omega_n(t))_{n\in\mathbb{N}}\subseteq\mathcal{L}^2\left([0,1],\mathcal{M}_{m\times d}\right)$ is relatively compact sequence of non-random weight functions. Then the sequence $(T_n)_{n\in\mathbb{N}}$ is asymptotically normal under $\mathbb{P}\in H_0$, and conditioned by the symmetric sub-σ-field $\mathcal{S}^0\left(\underline{X}_{k_n}\right)$. More explicitly,

$$\mathbb{P}*\left(T_n-E\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)\sim\mathcal{N}\left(0,Var\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)\ \left(\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right).$$

Proof: Let ξ_{ni}, $i=1,\,2,\,\cdots,\,k_n$, $n\in\mathbb{N}$, be defined by

$$\xi_{ni}:=\frac{1}{\sqrt{k_n}}\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right),\ 1\leqslant i\leqslant k_n,\ n\in\mathbb{N}.$$

From lemma 3.1.13, the random arrays $\xi_{n1},\ \cdots,\ \xi_{nk_n}$, are H_0-exchangeable under $\mathbb{P}\left(\cdot,\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)$. Let

$$F_n(t):=\sum_{i=1}^{[k_nt]}\frac{1}{\sqrt{k_n}}\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right),$$

$0\leqslant t\leqslant 1$, and let

$$F_n'(t):=F_n\left(\frac{i-1}{k_n}\right)+k_n\left(t-\frac{i-1}{k_n}\right)\left(F_n\left(\frac{i}{k_n}\right)-F_n\left(\frac{i-1}{k_n}\right)\right),$$

where $i:=[k_nt]+1$.
Therefore, we can write

$$T_n-E\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)=\int_0^1\omega_n(t)dF_n'(t).$$

Hence, all assumption of theorem 3.1.9 are satisfied, and consequently we have

$$\mathbb{P}*\left(T_n-E\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)\sim\mathcal{N}\left(0,\sigma_n^2\right)\ \left(\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right).$$

where $\sigma_n^2:=\int_0^1\omega_n(t)s^2\left(\underline{\xi}_{k_n}\right)\omega_n^t(t)dt-\int_0^1\omega_n(t)dts^2\left(\underline{\xi}_{k_n}\right)\int_0^1\omega_n^t(t)dt=$

$$=\int_0^1\omega_n(t)Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\omega_n^t(t)dt-\int_0^1\omega_n(t)dtVar\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\int_0^1\omega_n^t(t)dt.$$

Also, from remark 3.1.10 we can write

$$\sigma_n^2 = \frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni} s^2(\underline{\xi}_{k_n})\omega_{ni}^t - \frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni} s^2(\underline{\xi}_{k_n})\frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni}^t$$

$$\sigma_n^2 = \frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni} Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\omega_{ni}^t - \frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni} Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni}^t \ [\mathbb{P}]$$

By remarking that $\sigma_n^2 - Var\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right) \xrightarrow{\mathbb{P}} 0$, we conclude that the assertion is valid. □

Remark 3.1.16. From theorem 1.3.4, it is easy to see that the assertion of theorem 3.1.15 is equivalent to

$$\mathbb{P} * T_n \sim \mathcal{N}\left(E\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right), Var\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right) \ \left(\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right),$$

if the sequence $\left(E\left(T_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\right)_{n\in\mathbb{N}}$ is stochastically bounded.

3.2. The Hypothesis H_1 (symmetry):

Introduction: Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathbb{P}$ be a probability measure defined on $\mathcal{A}$, $\underline{X,Y}_{k_n} = ((X_{n1}, Y_{n1}), (X_{n2}, Y_{n2}), \cdots, (X_{nk_n}, Y_{nk_n}))^t$, $n \in \mathbb{N}$, be a triangular array of random couples with values in a sample space $(\mathcal{E}^2, \mathcal{B}^2)$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. Assume that the random couples of each row (i.e. $(X_{n1}, Y_{n1}), \cdots, (X_{nk_n}, Y_{nk_n})$, $n \in \mathbb{N}$) are i.i.d. under $\mathbb{P}$, and the distribution of each couple is symmetric. In other words, $(X_{ni}, Y_{ni}) \sim (Y_{ni}, X_{ni})$ for $i = 1, \cdots, k_n$. Let the distribution of (X_{ni}, Y_{ni}) under $\mathbb{P}$ be denoted by $\mathbb{P} * (X_{ni}, Y_{ni})$ or shortly by P_n. The general form of linear statistics which are considered here is denoted by $T_n\left(\underline{X,Y}_{k_n}\right)$, or shortly by T_n if we can avoid confusion, where $T_n := \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \omega_{ni} f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right)$, and where $\omega_{ni} \in \mathcal{M}_{m\times d}$, $i = 1, \cdots, k_n$, $n \in \mathbb{N}$, where the matrices ω_{ni}, $i = 1, \cdots, k_n$, $n \in \mathbb{N}$, are assumed to be given constants. We define here the following concept of symmetry
w depends on $\underline{x,y}_{k_n}$ in a permutation symmetric way iff

$$w\left(\pi\left(e\left(\underline{x,y}_{k_n}\right)\right)\right) = w\left(\underline{x,y}_{k_n}\right), \forall\ e \in \Xi_{k_n}, \pi \in \Pi_{k_n}.$$

We mention here that this is the symmetry concept, which is used in all arguments related with this section, and the symbols e, Ξ_{k_n} are introduced just before definition 3.2.1. The functions $f_n : \mathcal{E}^2 \times (\mathcal{E}^2)^{k_n} \longrightarrow \mathbb{R}^d$, $((x,y), \underline{z}_{k_n}) \longmapsto f_n((x,y), \underline{z}_{k_n})$, $n \in \mathbb{N}$, are measurable and such that they depend on $\underline{z}_{k_n}$ in a permutation symmetric way, and semi-symmetric one with respect to their first variable (x,y) i.e. $f_n((x,y), \underline{z}_{k_n}) = -f_n((y,x), \underline{z}_{k_n})$, $n \in \mathbb{N}$, and for all $(x,y) \in \mathcal{E}^2$. Fix $n \in \mathbb{N}$, and let $\lambda \in \mathbb{R}^d$ be a given constant, define $\Omega_\lambda \subseteq \Omega$ by

$$\Omega_\lambda := \Big\{\lambda^t f_n\left((X_{n1}, Y_{n1}), \underline{X,Y}_{k_n}\right) = \lambda^t f_n\left((X_{n2}, Y_{n2}), \underline{X,Y}_{k_n}\right) = \cdots$$
$$\cdots = \lambda^t f_n\left((X_{nk_n}, Y_{nk_n}), \underline{X,Y}_{k_n}\right)\Big\}$$

we assume that $\mathbb{P}(\Omega_\lambda) > 0$ is hold iff $\lambda = 0$. Now, for notational convenience concerning the computations, let $\omega_n \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d})$ be such that

$\omega_{ni} = k_n \int_{\frac{i-1}{k_n}}^{\frac{i}{k_n}} \omega_n(t)dt$, $i = 1, \cdots, k_n$, $n \in \mathbb{N}$.

We mention here that the form of the considered statistic T_n contains the linear rank test statistic $S = \sum_{i=1}^{n} a(R_i^+)\mathtt{sign}X_i$ which has been discussed deeply in "Theory of rank tests [1967]" which is due to J. Hájek and Z. Šidák.

To achieve the purpose here we need to build a base for our coming limit theorems, for this we shall begin with the following interesting steps which are needed very much to obtain the final results.

Let us here introduce a new kind of exchangeablity, we call it H_1-exchangeablity, also we put some related symbols, which will be needed for the main results of this section.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let

$$\underline{\xi}_{k_n} = (\xi_{n1}, \cdots, \xi_{nk_n})^t,\ \underline{\tilde{\xi}}_{k_n} = (\tilde{\xi}_{n1}, \cdots, \tilde{\xi}_{nk_n})^t$$

be triangular arrays of random elements from $(\Omega, \mathcal{A}, P)$ to $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. let us define $\underline{\xi, \tilde{\xi}}_{k_n} := \left((\xi_{n1}, \tilde{\xi}_{n1}), \cdots, (\xi_{nk_n}, \tilde{\xi}_{nk_n})\right)$, and let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence of sub-σ-fields of $\mathcal{A}$, $\mathcal{C}$ be a sub-σ-field of $\mathcal{A}$. Let further $\mathcal{S}^1\left(\underline{\xi, \tilde{\xi}}_{k_n}\right) := \left(\underline{\xi, \tilde{\xi}}_{k_n}\right)^{-1}(\mathcal{S}_n^1)$, where $\mathcal{S}_n^1$ denotes here the σ-field of all sets B in $(\mathcal{B}^2)^{k_n}$ satisfying the condition

$$\underline{x, \tilde{x}}_{k_n} \in B \Longleftrightarrow \pi\left(e\left(\underline{x, \tilde{x}}_{k_n}\right)\right) \in B,\ \forall \pi \in \Pi_{k_n},\ \forall e \in \Xi_{k_n}$$

where here $\underline{x, \tilde{x}}_{k_n} := ((x_1, \tilde{x}_1), \cdots, (x_{k_n}, \tilde{x}_{k_n}))$, and $\Xi_{k_n} := \Big\{e : e\left(\underline{x, \tilde{x}}_{k_n}\right) = (e_1(x_1, \tilde{x}_1), \cdots, e_n(x_{k_n}, \tilde{x}_{k_n})),\ e_i = id,\ \text{or}\ e_i(x, \tilde{x}) = (\tilde{x}, x),\ \forall\, \underline{x, \tilde{x}}_{k_n} \in (\mathcal{E}^2)^{k_n},\ i = 1, \cdots, n\Big\}$. Let $s^2(\underline{\xi}_{k_n}) := \sum_{i=1}^{k_n} \xi_{ni}\xi_{ni}^t$. For each $n \in \mathbb{N}$ the matrix $s^2(\underline{\xi}_{k_n})$ is assumed to be positive definite. All these assumptions are hold in this section.

Definition 3.2.1. The random pairs $(\xi_{n1}, \tilde{\xi}_{n1}), \cdots, (\xi_{nk_n}, \tilde{\xi}_{nk_n})$ are H_1-exchangeable under $P(\cdot|\mathcal{C}_n)$, iff $\forall \pi \in \Pi_{k_n},\ \forall e \in \Xi_{k_n}$ the equality

$$P\left(\underline{\xi, \tilde{\xi}}_{k_n} \in B|\mathcal{C}_n\right) = P\left(\pi e\left(\underline{\xi, \tilde{\xi}}_{k_n}\right) \in B|\mathcal{C}_n\right)\ [P]$$

is valid for all $B \in (\mathcal{B}^2)^{k_n}$, and then the array $\underline{\xi, \tilde{\xi}}_{k_n}$ is called H_1-exchangeable under $P(\cdot|\mathcal{C}_n)$.

Lemma 3.2.2. Let $f : (\mathcal{E}^2, \mathcal{B}^2)^{k_n} \longrightarrow (\mathbb{R}, \mathbb{B})$ be a measurable function such that $E(f(\underline{\xi, \tilde{\xi}}_{k_n}))$ is well-defined. If $\underline{\xi, \tilde{\xi}}_{k_n}$ is H_1-exchangeable under $P(\cdot|\mathcal{C})$, then $\forall A \in \mathcal{S}^1(\underline{\xi, \tilde{\xi}}_{k_n}),\ C \in \mathcal{C},\ \pi \in \Pi_{k_n},\ e \in \Xi_{k_n}$ the following equality is valid

$$E\left(1_{A\cap C} f\left(\pi\left(e\left(\underline{\xi, \tilde{\xi}}_{k_n}\right)\right)\right)\right) = E\left(1_{A\cap C} f\left(\underline{\xi, \tilde{\xi}}_{k_n}\right)\right).$$

□

Let us denote the σ-field $\sigma(\mathcal{S}^1\left(\underline{\xi, \tilde{\xi}}_{k_n}\right), \mathcal{C})$ by $\mathcal{S}^1(\underline{\xi, \tilde{\xi}}_{k_n}, \mathcal{C})$. Now, since $\{A \cap C : A \in \mathcal{S}^1\left(\underline{\xi, \tilde{\xi}}_{k_n}\right),\ C \in \mathcal{C}\}$ generates the σ-field $\mathcal{S}^1(\underline{\xi, \tilde{\xi}}_{k_n}, \mathcal{C})$, we have by lemma 3.2.2 the validity of

$$E\left(1_D f\left(\pi\left(e\left(\underline{\xi, \tilde{\xi}}_{k_n}\right)\right)\right)\right) = E\left(1_D f\left(\underline{\xi, \tilde{\xi}}_{k_n}\right)\right),$$

$\forall D \in \mathcal{S}^1(\underline{\xi, \tilde{\xi}}_{k_n}, \mathcal{C}),\ \pi \in \Pi_{k_n},\ e \in \Xi_{k_n}$.
Therefore, the following lemma is just a consequence of lemma 3.2.2.

Lemma 3.2.3. Let $f : (\mathcal{E}^2, \mathcal{B}^2)^{k_n} \longrightarrow (\mathbb{R}, \mathbb{B})$ be a measurable function such that $E(f(\underline{\xi, \tilde{\xi}}_{k_n}))$ is well-defined. If $\underline{\xi, \tilde{\xi}}_{k_n}$ is H_1-exchangeable under $P(\cdot|\mathcal{C})$, then

$$E(f(\underline{\xi, \tilde{\xi}}_{k_n})|\mathcal{S}^1\left(\underline{\xi, \tilde{\xi}}_{k_n}, \mathcal{C}\right)) = \frac{1}{2^{k_n} k_n!} \sum_{\pi \in \Pi_{k_n}, e \in \Xi_{k_n}} f(\pi(e(\underline{\xi, \tilde{\xi}}_{k_n})))\ [P].$$

Now, we are at the suitable position to introduce the following invariance principle which is the base, needed for building the limit theorems of this section.

Theorem 3.2.4. Suppose that the triangular array $\underline{\xi}_{k_n}$ is infinitesimal, and the sequence $\left(s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$ is stochastically bounded. If $\mathcal{C}_n \supseteq \mathcal{S}^1\left(\underline{\xi, -\xi}_{k_n}\right)$, $n \in \mathbb{N}$, is a sequence of sub-σ-fields of $\mathcal{A}$, $\tilde{\mathcal{C}}_n := \sigma(C \times$

$(C[0,1])^d : C \in \mathcal{C}_n)$, and if the triangular array $\underline{\xi, -\xi}_{k_n}$ is H_1-exchangeable under $P(\cdot|\mathcal{C}_n)$ then

$$\tilde{\mathcal{S}}_n \overset{w}{\sim} \tilde{W}\ (\tilde{\mathcal{C}}_n).$$

Where

$$\tilde{\mathcal{S}}_n : \Omega \times (C[0,1])^d \longrightarrow (D[0,1])^d,$$
$$\tilde{\mathcal{S}}_n(\omega_1, \omega_2) := \mathcal{S}_n(\omega_1),$$
$$(\mathcal{S}_n(\omega_1))(t) := \sum_{i=1}^{[k_n t]} \xi_{ni}(\omega_1).$$

And

$$\tilde{W} : \Omega \times (C[0,1])^d \longrightarrow (C[0,1])^d \subset (D[0,1])^d,$$
$$\tilde{W}(\omega_1, \omega_2) := \left(\sqrt{s^2(\underline{\xi}_{k_n})}\right)(\omega_1) W(\omega_2),$$

where $W := (W_t)_{0 \leqslant t \leqslant 1}$ is a d-dimensional Brownian motion.

Proof: We first prove it in the case where $\sup_n \| s^2(\underline{\xi}_{k_n}) \| < +\infty\ [P]$.
By lemma 1.2.2 the assertion is equivalent to the validity of the limit

$$\int\limits_{C_n \times C[0,1]^d} \varphi(\tilde{\mathcal{S}}_n) dP \otimes \mu - \int\limits_{C_n \times C[0,1]^d} \varphi(\tilde{W}) dP \otimes \mu \underset{n \longrightarrow \infty}{\longrightarrow} 0,$$

for all sequences $(C_n)_{n\in\mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where $C_n \in \mathcal{C}_n$, and where μ is the Wiener measure defined on $(C[0,1])^d$.
Therefore, let $(C_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of the measurable sets $C_n \in \mathcal{C}_n$, $n \in \mathbb{N}$, we want to prove the validity of the previous limit above, which is equivalent to

$$\int\limits_{C_n} \varphi(\mathcal{S}_n) dP - \int\limits_{C_n} \int\limits_{(C[0,1])^d} \varphi(\tilde{W}) dP d\mu \underset{n \longrightarrow \infty}{\longrightarrow} 0$$

$$\Longleftrightarrow \int\limits_{C_n} E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right) dP - \int\limits_{C_n} E\left(\int\limits_{(C[0,1])^d} \varphi(\tilde{W}) d\mu \middle| \mathcal{C}_n\right) dP \underset{n \longrightarrow \infty}{\longrightarrow} 0$$

Since the sequence $(C_n)_{n\in\mathbb{N}}$ is arbitrary, the previous assertion is equivalent to

$$E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right) - E\left(\int\limits_{(C[0,1])^d} \varphi(\tilde{W}) d\mu \middle| \mathcal{C}_n\right) \overset{P}{\longrightarrow} 0$$

$$\Longleftrightarrow E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right) - \int\limits_{(C[0,1])^d} \varphi(\tilde{W})d\mu \xrightarrow{P} 0$$

$$\Longleftrightarrow \frac{1}{2^{k_n}k_n!} \sum_{\substack{\pi \\ \varepsilon_1,\cdots,\varepsilon_{k_n}\in\{-1,+1\}}} \varphi\left(\sum_{i=1}^{[k_n(\cdot)]} \varepsilon_i \xi_{n\pi(i)}\right) - \int\limits_{(C[0,1])^d} \varphi(\tilde{W})d\mu \xrightarrow{P} 0,$$

where π here ranging over all bijective functions, defined on $\{1, 2, \cdots, k_n\}$. It is sufficient to prove that each subsequence $\{n'\}$ contains another sub-subsequence $\{n''\}$ such that

$$\frac{1}{2^{k_{n''}}k_{n''}!} \sum_{\substack{\pi \\ \varepsilon_1,\cdots,\varepsilon_{k_{n''}}\in\{-1,+1\}}} \varphi\left(\sum_{i=1}^{[k_{n''}(\cdot)]} \varepsilon_i \xi_{n''\pi(i)}\right) - \int\limits_{(C[0,1])^d} \varphi(\tilde{W})d\mu \longrightarrow 0 \ [P].$$

From the hypotheses, each sub-sequence $\{n'\}$ of $\{n\}$ contains another sub-subsequence $\{n''\}$ such that $\max\limits_{1\leqslant i\leqslant k_{n''}} \parallel \xi_{n''i} \parallel \longrightarrow 0 \ [P]$. Therefore, without any loss of generality we can assume that $\max\limits_{1\leqslant i\leqslant k_n} \parallel \xi_{ni} \parallel \longrightarrow 0 \ [P]$. Hence, for all ω_1 of those fulfill the hypotheses, we want now to prove

$$\frac{1}{2^{k_n}k_n!} \sum_{\substack{\pi \\ \varepsilon_1,\cdots,\varepsilon_{k_n}\in\{-1,+1\}}} \varphi\left(\sum_{i=1}^{[k_n(\cdot)]} \varepsilon_i \xi_{n\pi(i)}(\omega_1)\right) - $$
$$- \int\limits_{(C[0,1])^d} \varphi\left(\left(\sqrt{s^2(\underline{\xi}_{k_n})}\right)(\omega_1)W\right) d\mu \longrightarrow 0.$$

But this is valid indeed from remark 2.3.4.
Now, we turn to prove it in the general case.
Let us define $C_n^M := \left\{ \parallel s^2(\underline{\xi}_{k_n}) \parallel \leqslant M \right\}$, and also for each $n \in \mathbb{N}$, and $i = 1, \cdots, k_n$, we define the new random arrays $\tilde{\xi}_{ni} := 1_{C_n^M} \cdot \xi_{ni}$. It is clear that the new random arrays satisfy the first case of the proof above.
Consequently, the limit

$$\int\limits_{C_n\times(C[0,1])^d} \varphi(1_{C_n^M} \cdot \tilde{\mathcal{S}}_n)dP\otimes\mu - \int\limits_{C_n\times(C[0,1])^d} \varphi(1_{C_n^M} \cdot \tilde{W})dP\otimes\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

is valid for all sequences $(C_n)_{n\in\mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where $C_n \in \mathcal{C}_n$. We can rewrite the previous limit as the following

$$\int\limits_{C_n\times(C[0,1])^d} 1_{C_n^M} \cdot \varphi(\tilde{\mathcal{S}}_n)dP\otimes\mu - \int\limits_{C_n\times(C[0,1])^d} 1_{C_n^M} \cdot \varphi(\tilde{W})dP\otimes\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

Also, we can enlarge M to make the following inequality valid $P(C_n^M) \geqslant 1-\varepsilon$ for any given $\varepsilon > 0$, and for all $n \in \mathbb{N}$.
Therefore, we obtain the validity of the limit

$$\int\limits_{C_n \times (C[0,1])^d} \varphi(\tilde{\mathcal{S}}_n) dP \otimes \mu - \int\limits_{C_n \times (C[0,1])^d} \varphi(\tilde{W}) dP \otimes \mu \underset{n \longrightarrow \infty}{\longrightarrow} 0,$$

for all sequences $(C_n)_{n \in \mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where $C_n \in \mathcal{C}_n$.
Hence, the proof is complete. □

Corollary 3.2.5. In the same situation of theorem 3.2.4, we have the validity of

$$\tilde{\mathcal{S}}_n \overset{w}{\sim} \tilde{W},$$

or more explicitly

$$\left(\sum_{i=1}^{[k_n t]} \xi_{ni} \right)_{0 \leqslant t \leqslant 1} \overset{w}{\sim} \left(\sqrt{s^2(\underline{\xi}_{k_n})} W_t \right)_{0 \leqslant t \leqslant 1}.$$

Remark 3.2.6. Let X_n be defined by $\begin{pmatrix} S_n(t_1) \\ \vdots \\ S_n(t_k) \end{pmatrix}$, where $t_1,\ t_2, \cdots,$ t_k belong to $[0,1]$, and let $a_1,\ a_2, \cdots,\ a_k$ be given constant matrices of order $m \times d$, then we have
$P * \sum\limits_{i=1}^{k} a_i \cdot S_n(t_i) \sim \mathcal{N}\left(\mu_n, \sigma_n^2\right)\ (\mathcal{C}_n)$, where $\forall n \in \mathbb{N}$

$$\mu_n := 0$$

$$\sigma_n^2 := (a_1, \cdots, a_k) \cdot \left(\min(t_i, t_j) s^2 \left(\underline{\xi}_{k_n} \right) \right)_{1 \leqslant i,j \leqslant k} \cdot \begin{pmatrix} a_1^t \\ \vdots \\ a_k^t \end{pmatrix}$$

Proof: From Theorem 3.2.4 we conclude the following fact

$$P * \begin{pmatrix} S_n(t_1) \\ \vdots \\ S_n(t_k) \end{pmatrix} \sim \mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right)\ (\mathcal{C}_n),$$

where here $\forall n \in \mathbb{N}$

$$\tilde{\mu}_n := 0,$$

$$\tilde{\sigma}_n^2 := \left(\min(t_i, t_j) s^2(\underline{\xi}_{k_n})\right)_{1\leqslant i,j\leqslant k}.$$

Which means by definition the validity of the following limit $E_P\left(f\left(S_n(t_1), S_n(t_2), \cdots, S_n(t_k)\right)|\mathcal{C}_n\right) - \int f d\mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right) \xrightarrow{P} 0$, for all bounded uniformly continuous functions $f : (\mathbb{R}^d)^k \longrightarrow \mathbb{R}$. Now, let $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a bounded uniformly continuous function, and let $a_1, a_2, \cdots, a_k \in \mathcal{M}_{m\times d}$ be given constant matrices, then the function $h : (\mathbb{R}^d)^k \longrightarrow \mathbb{R}$, defined by $h(x) := g((a_1, a_2, \cdots, a_k)\cdot x)$, $\forall x \in (\mathbb{R}^d)^k$ is also bounded uniformly continuous function. Thus, we have for this h the validity of the limit $E_P\left(h\left(S_n(t_1), S_n(t_2), \cdots, S_n(t_k)\right)|\mathcal{C}_n\right) - \int h d\mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right) \xrightarrow{P} 0$, which can be rewritten as

$$E_P\left(g\left((a_1, a_2, \cdots, a_k)\cdot \begin{pmatrix} S_n(t_1) \\ \vdots \\ S_n(t_k)\end{pmatrix}\right)\Bigg|\mathcal{C}_n\right) - \int g d\mathcal{N}\left(\mu_n, \sigma_n^2\right) \xrightarrow{P} 0,$$ which is also equivalent to

$$E_P\left(g\left(\sum_{j=1}^{k} a_j \cdot S_n(t_j)\right)|\mathcal{C}_n\right) - \int g d\mathcal{N}\left(\mu_n, \sigma_n^2\right) \xrightarrow{P} 0,$$

where here

$$\mu_n := (a_1, a_2, \cdots, a_k)\cdot \begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix} = 0,$$

and

$$\sigma_n^2 := (a_1, \cdots, a_k)\cdot \left(\min(t_i, t_j) s^2\left(\underline{\xi}_{k_n}\right)\right)_{1\leqslant i,j\leqslant k} \cdot \begin{pmatrix} a_1^t \\ \vdots \\ a_k^t\end{pmatrix}.$$

Therefore, the proof is complete. □

Corollary 3.2.7. In remark 3.2.6 if $0 < t_1 < t_2 < \cdots < t_k < 1$, then we can rewrite σ_n^2 as the following

$$\sigma_n^2 = \sum_{j=1}^{k}\left(\sum_{i=1}^{j} a_i s^2(\underline{\xi}_{k_n}) a_j^t t_i + \sum_{i=1}^{k-j} a_{j+i} s^2(\underline{\xi}_{k_n}) a_j^t t_j\right).$$

Also, if $t = \frac{1}{k}$, $a_j = \alpha_j - \alpha_{j+1}$, and $a_k = \alpha_k$, where α_1, α_2,$\cdots$, α_k, are given constant matrices of order $m \times d$, and if $t_j = jt$, for $j = 1, 2, \cdots, k$, then we have easily
$\mu_n = 0$, and
$\sigma_n^2 = \left(\sum_{j=1}^{k} \alpha_j s^2 \left(\underline{\xi}_{k_n} \right) \alpha_j^t \right) t$. These results are involved in the proof of theorem 3.2.9. □

Throughout the proofs of the coming limit theorems in this section we need to represent some Partial sums as path integrals. Let ξ_{n1}, ξ_{n2},$\cdots$,ξ_{nk_n} be random arrays from $(\Omega, \mathcal{A}, P)$ to $(\mathbb{R}^d, \mathbb{B}^d)$, $S_n(t) := \sum_{i=1}^{[k_n t]} \xi_{ni}$, and let ω_{ni}, $i = 1, 2, \cdots, k_n$, $n \in \mathbb{N}$, be constant matrices of order $m \times d$. One can write $\sum_{i=1}^{k_n} \omega_{ni} \xi_{ni} = \int_0^1 \omega_n(t) dS'(t)$, where $S'_n(t)$ is defined by $S'_n(t) = S_n\left(\frac{i-1}{k_n}\right) + k_n \left(t - \frac{i-1}{k_n}\right) \left(S_n\left(\frac{i}{k_n}\right) - S_n\left(\frac{i-1}{k_n}\right)\right)$, where $i := [k_n t] + 1$ in this formula, and $\omega_n \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d})$ is defined such that $\omega_{ni} = k_n \int_{\frac{i-1}{k_n}}^{\frac{i}{k_n}} \omega_n(t) dt$, for $i = 1, \cdots, k_n$. Also, we need to see the behavior of $\| \sum_{i=1}^{k_n} \omega_{ni} \xi_{ni} \|$ when $\| \omega_n \|_2$ tends to zero with n tends to infinity, the next lemma gives an answer to that question.

Lemma 3.2.8. If the random arrays ξ_{n1}, ξ_{n2},$\cdots$,ξ_{nk_n} are H_1-exchangeable under $P(\cdot|\mathcal{C}_n)$, and the sequence $\left(s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$ is stochastically bounded, then

$$\| \sum_{i=1}^{k_n} \omega_{ni} \xi_{ni} \| \xrightarrow{P} 0, \text{ if } \| \omega_n \|_2 \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

Proof: Since $\omega \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d})$, and from Lemma 3.2.3, we have
$E\left(\left(\sum_{i=1}^{k_n} \omega_{ni} \xi_{ni} \right) \left(\sum_{i=1}^{k_n} \omega_{ni} \xi_{ni} \right)^t \middle| \mathcal{S}^1 \left(\underline{\xi}_{k_n}, \mathcal{C}_n \right) \right) = \frac{1}{k_n} \sum_{i=1}^{k_n} \omega_{ni} s^2(\underline{\xi}_{k_n}) \omega_{ni}^t.$
Now, since

$$\frac{1}{k_n} \sum_{i=1}^{k_n} \| \omega_{ni} \|^2 \leqslant \int_0^1 \| \omega_n \|^2 \, dt = \| \omega_n \|_2^2,$$

and since the sequence $\left(s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$ is stochastically bounded, we obtain
$E\left(\left(\sum\limits_{i=1}^{k_n}\omega_{ni}\xi_{ni}\right)\left(\sum\limits_{i=1}^{k_n}\omega_{ni}\xi_{ni}\right)^t \middle| \mathcal{S}^1\left(\underline{\xi}_{k_n},\mathcal{C}_n\right)\right) \xrightarrow{P} 0.$
Therefore,

$$E\left(\parallel \sum_{i=1}^{k_n}\omega_{ni}\xi_{ni}\parallel^2 \middle| \mathcal{S}^1\left(\underline{\xi}_{k_n},\mathcal{C}_n\right)\right) \xrightarrow{P} 0,$$

and from theorem 1.1.3 we conclude that the assertion is valid indeed. □

In the next theorem we discuss the asymptotic normality of the sum $\sum\limits_{i=1}^{k_n}\omega_{ni}\xi_{ni}$.

Theorem 3.2.9. Suppose that the triangular array $\underline{\xi}_{k_n}$ is infinitesimal and that the sequence $\left(s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$ is stochastically bounded, and further that sequence $(\omega_n)_{n\in\mathbb{N}}$ is a relatively compact sequence in the space $\mathcal{L}^2([0,1],\mathcal{M}_{m\times d})$. We call it a sequence of weight functions and assume that these weights are constants. If $\mathcal{C}_n \supseteq \mathcal{S}^1\left(\underline{\xi}_{k_n}\right)$, $n\in\mathbb{N}$, is a sequence of sub-σ-fields of $\mathcal{A}$, and if the rows of the triangular array are H_1-exchangeable under $P(\cdot|\mathcal{C}_n)$, then we have

$$P * \sum_{i=1}^{k_n}\omega_{ni}\xi_{ni} \sim \mathcal{N}(\mu_n,\sigma_n^2)\ (\mathcal{C}_n),$$

where

$$\mu_n := 0,$$

and

$$\sigma_n^2 := \int_0^1 \omega_n(t)s^2(\underline{\xi}_{k_n})\omega_n^t(t)dt.$$

Proof: Since, $(\omega_n(t))_{n\in\mathbb{N}}$ is relatively compact, so every subsequence of $(\omega_n(t))_{n\in\mathbb{N}}$ contains another convergent sub-subsequence, and consequently it is sufficient to prove the assertion in the case when the sequence $(\omega_n(t))_{n\in\mathbb{N}}$ is convergent. And from lemma 3.2.8 we conclude that it is sufficient to prove it in the case $\omega_n(t) = \omega(t)$, $\forall n\in\mathbb{N}$. Moreover, remarking that any $\omega\in\mathcal{L}^2([0,1],\mathcal{M}_{m\times d})$ can assumed as a limit of some sequence of step-functions belong to $M_{m\times d}$, where $M_{m\times d} := \Big\{\omega : \omega(t) = \sum\limits_{i=1}^{k} x_i\cdot 1_{[\frac{i-1}{k},\frac{i}{k})}(t), x_i \in$

$\mathcal{M}_{m\times d},\ i=1,\cdots,k,\ k\in\mathbb{N}\Big\}$. By applying lemma 3.2.8 we find that it is sufficient to prove it when $\omega\in M_{m\times d}$. Now, we note that
$\sum_{i=1}^{k_n}\omega_{ni}\xi_{ni}=\int_0^1\omega(t)dS'_n(t)=\sum_{i=1}^{k}x_i\left(S'_n\left(\frac{i}{k}\right)-S'_n\left(\frac{i-1}{k}\right)\right)=\sum_{i=1}^{k}a_iS'_n\left(\frac{i}{k}\right)$, where $a_i:=x_i-x_{i+1}$, $i=1,2,\cdots,k-1$, $a_k=x_k$. Since the triangular array $\underline{\xi}_{k_n}$ is infintesimal, and from remark 3.2.6, and corollary 3.2.7, we find that the assertion is valid indeed. □

Remark 3.2.10. In theorem 3.2.9, σ_n^2 can be rewritten as
$\sigma_n^2=\frac{1}{k_n}\sum_{i=1}^{k_n}\omega_{ni}s^2(\underline{\xi}_{k_n})\omega_{ni}^t$, $n\in\mathbb{N}$. □

We want at this position to introduce some important properties of the concepts of the H_1-exchangeablity under $P(\cdot|\mathcal{C})$, where $\mathcal{C}$ is a given sub-σ-field of $\mathcal{A}$, also we want to discuss some properties of the σ-field $\mathcal{S}^1\left(\underline{\xi,\tilde{\xi}}_k\right)$, $k\in\mathbb{N}$. For this, we shall introduce the definition of the H_1-exchangeablity under P, also some interesting results.

Definition 3.2.11. Under the same hypotheses of definition 3.2.1, we call the triangular array $\underline{\xi,\tilde{\xi}}_{k_n}$ H_1-exchangeable under P iff this triangular array is H_1-exchangeable under $P(\cdot|\mathcal{C})$, for $\mathcal{C}=\{\emptyset,\Omega\}$.

– It is clear from lemma 3.2.3 that the H_1-exchangeablity under $P(\cdot|\mathcal{C})$, for all $\mathcal{C}\subseteq\mathcal{S}^1\left(\underline{\xi,\tilde{\xi}}_{k_n}\right)$ is equivalent to the H_1-exchangeablity under P.

Lemma 3.2.12. Let X_1, X_2, $\cdots$, X_k, Y_1, Y_2, $\cdots$, Y_k be random elements from $(\Omega,\mathcal{A},P)$ to $(\mathcal{E},\mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. Let further, $f:\mathcal{E}^2\times(\mathcal{E}^2)^k\longrightarrow\mathbb{R}^m$ be a measurable function, which is permutation symmetric in its second variable, and semi-symmetric in its first variable, and let $\xi_i:=f\left((X_i,Y_i),\underline{X,Y}_k\right)$, $i=1,\cdots,k$, then $\mathcal{S}^1\left(\underline{\xi,-\xi}_k\right)\subseteq\mathcal{S}^1\left(\underline{X,Y}_k\right)$.

Proof: It is sufficient to prove that the following implication
$g\left(\underline{\xi,-\xi}_k\right)$ is $\mathcal{S}^1\left(\underline{\xi,-\xi}_k\right)$-measurable $\Longrightarrow$

$$\Longrightarrow g\left(\underline{\xi,-\xi}_k\right)\text{ is }\mathcal{S}^1\left(\underline{X,Y}_k\right)\text{-measurable}$$

is valid for all measurable $g : \left((\mathbb{R}^m)^2\right)^k \longrightarrow \mathbb{R}$. For this, let $h : (\mathcal{E}^2)^k \longrightarrow \left((\mathbb{R}^m)^2\right)^k$ be defined by

$$h(\underline{x,y}_k) := \Big(\Big(f\left((x_1,y_1), \underline{x,y}_k\right), -f\left((x_1,y_1), \underline{x,y}_k\right)\Big), \cdots , \Big(f\left((x_k,y_k), \underline{x,y}_k\right), -f\left((x_k,y_k), \underline{x,y}_k\right)\Big)\Big),$$

where $\underline{x,y}_k = ((x_1,y_1), \cdots, (x_k,y_k)) \in (\mathcal{E}^2)^k$. Let $g\left(\underline{\xi,-\xi}_k\right)$ be $\mathcal{S}^1\left(\underline{\xi,-\xi}_k\right)$-measurable, this implies $g\left(\pi\left(e\left(\underline{\xi,-\xi}_k\right)\right)\right) = g\left(\underline{\xi,-\xi}_k\right)$ for all $\pi \in \Pi_k$, and all $e \in \Xi_k$. Let us define $g_0 : (\mathcal{E}^2)^k \longrightarrow \mathbb{R}$ by $g_0\left(\underline{x,y}_k\right) := g(h(\underline{x,y}_k))$.
Hence, $g_0\left(\underline{X,Y}_k\right) = g\left(h\left(\underline{X,Y}_k\right)\right) = g\left(\underline{\xi,-\xi}_k\right)$, and $g_0\left(\pi\left(e\left(\underline{X,Y}_k\right)\right)\right)$ $= g\left(h\left(\pi\left(e\left(\underline{X,Y}_k\right)\right)\right)\right) = g\left(\pi\left(e\left(\underline{\xi,-\xi}_k\right)\right)\right) = g\left(\underline{\xi,-\xi}_k\right) = g_0\left(\underline{X,Y}_k\right)$.
Hence, $g_0\left(\pi\left(e\left(\underline{X,Y}_k\right)\right)\right) = g_0\left(\underline{X,Y}_k\right)$, but this means that $g_0\left(\underline{X,Y}_k\right)$ is $\mathcal{S}^1\left(\underline{X,Y}_k\right)$-measurable, and also $g\left(\underline{\xi,-\xi}_k\right)$ is $\mathcal{S}^1\left(\underline{X,Y}_k\right)$-measurable. Consequently, the assertion is valid indeed. □

Lemma 3.2.13. Let $f : \mathcal{E}^2 \times (\mathcal{E}^2)^k \longrightarrow \mathbb{R}^m$ be a measurable function which is permutation symmetric in its second variable, and semi-symmetric in its first variable. If the random elements (X_1,Y_1), $\cdots$, (X_k,Y_k) are H_1-exchangeable under $P(\cdot|\mathcal{C})$, then $f\left((X_1,Y_1), \underline{X,Y}_k\right)$, $\cdots$, $f\left((X_k,Y_k), \underline{X,Y}_k\right)$, are also H_1-exchangeable under $P(\cdot|\mathcal{C})$.

Proof: It is sufficient to prove that for any measurable function $g : \left((\mathbb{R}^m)^2, (\mathbb{B}^m)^2\right)^k \longrightarrow (\mathbb{R}, \mathbb{B})$ satisfying that the quantity

$$E\Big(g\Big(\Big(f\left((X_1,Y_1), \underline{X,Y}_k\right), -f\left((X_1,Y_1), \underline{X,Y}_k\right)\Big), \cdots , \Big(f\left((X_k,Y_k), \underline{X,Y}_k\right), -f\left((X_k,Y_k), \underline{X,Y}_k\right)\Big)\Big)\Big)$$

is well-defined, the equality

$$E\Big(g\Big(\pi\Big(e\Big(\Big(f\left((X_1,Y_1), \underline{X,Y}_k\right), -f\left((X_1,Y_1), \underline{X,Y}_k\right)\Big), \cdots , \Big(f\left((X_k,Y_k), \underline{X,Y}_k\right), -f\left((X_k,Y_k), \underline{X,Y}_k\right)\Big)\Big)\Big)\Big) |\mathcal{C}\Big) =$$

$$= E\Big(g\Big(\Big(f\left((X_1,Y_1), \underline{X,Y}_k\right), -f\left((X_1,Y_1), \underline{X,Y}_k\right)\Big), \cdots , \Big(f\left((X_k,Y_k), \underline{X,Y}_k\right), -f\left((X_k,Y_k), \underline{X,Y}_k\right)\Big)\Big) |\mathcal{C}\Big) \quad [P]$$

is valid. For this, let us define $h : (\mathcal{E}^2)^k \longrightarrow (\mathbb{R}^m)^k$ by

$$h(\underline{x,y}_k) = \Big(\Big(f\left((x_1,y_1),\underline{x,y}_k\right), -f\left((x_1,y_1),\underline{x,y}_k\right)\Big),\cdots$$
$$,\Big(f\left((x_k,y_k),\underline{x,y}_k\right), -f\left((x_k,y_k),\underline{x,y}_k\right)\Big)\Big)^t,$$

where $\underline{x,y}_k = ((x_1,y_1),\cdots,(x_k,y_k)) \in (\mathcal{E}^2)^k$. And we can write

$$h(\underline{X,Y}_k) = \Big(\Big(f\left((X_1,Y_1),\underline{X,Y}_k\right), -f\left((X_1,Y_1),\underline{X,Y}_k\right)\Big),\cdots$$
$$,\Big(f\left((X_k,Y_k),\underline{X,Y}_k\right), -f\left((X_k,Y_k),\underline{X,Y}_k\right)\Big)\Big)^t.$$

Consequently, it remains to be proved the validity of the equality

$$E\left(g\left(\pi\left(e\left(h\left(\underline{X,Y}_k\right)\right)\right)\right)|\mathcal{C}\right) = E\left(g\left(h\left(\underline{X,Y}_k\right)\right)|\mathcal{C}\right)\ [P].$$

But easily, we see that the left side is equal to $E\left(g\left(h\left(\pi\left(e\left(\underline{X,Y}_k\right)\right)\right)\right)|\mathcal{C}\right)$ almost sure with respect to P. Since (X_1,Y_1), $\cdots$, (X_k,Y_k) are H_1-exchangeable under $P(\cdot|\mathcal{C})$, the assertion follows. □

Lemma 3.2.14. Let $\xi_1, \cdots, \xi_k$ be random arrays from $(\Omega,\mathcal{A},P)$ to $(\mathbb{R}^m,\mathbb{B}^m)$, and $\mathcal{C}\subseteq\mathcal{A}$ be a sub-σ-field. Let $\underline{x}_k = (x_1,\cdots,x_k) \in (\mathbb{R}^m)^k$, and $V(\underline{x}_k) = \sum_{i=1}^{k} x_i x_i^t$. Assume that $V(\underline{x}_k)$ is positive definite, and let $g : (\mathbb{R}^m)^2 \times \left((\mathbb{R}^m)^2\right)^k \longrightarrow \mathbb{R}^m$ be defined by $g\left((y,\tilde{y}),\underline{x,\tilde{x}}_k\right) = \left(V\left(\underline{\frac{x-\tilde{x}}{2}}_k\right)\right)^{-\frac{1}{2}}\left(\frac{y-\tilde{y}}{2}\right)$, and $\eta_i := \left(V\left(\underline{\xi}_k\right)\right)^{-\frac{1}{2}}\xi_i = g\left((\xi_i,-\xi_i),\underline{\xi,-\xi}_k\right)$, $i=1,\cdots,k$. If $(\xi_1,-\xi_1)$, $\cdots$, $(\xi_k,-\xi_k)$ are H_1-exchangeable under $P(\cdot|\mathcal{C})$, and $V\left(\underline{\xi}_k\right)$ is $\mathcal{C}$-measurable, then $\mathcal{S}^1\left(\underline{\xi,-\xi}_k,\mathcal{C}\right) = \mathcal{S}^1\left(\underline{\eta,-\eta}_k,\mathcal{C}\right)$.

Proof: First we see that $\mathcal{S}^1\left(\underline{\eta,-\eta}_k,\mathcal{C}\right) \subseteq \mathcal{S}^1\left(\underline{\xi,-\xi}_k,\mathcal{C}\right)$ follows from lemma 3.2.12. It remains to show that $\mathcal{S}^1\left(\underline{\xi,-\xi}_k,\mathcal{C}\right) \subseteq \mathcal{S}^1\left(\underline{\eta-\eta}_k,\mathcal{C}\right)$. And for this it is sufficient to prove that $\mathcal{S}^1\left(\underline{\xi,-\xi}_k\right) \subseteq \mathcal{S}^1\left(\underline{\eta,-\eta}_k,\mathcal{C}\right)$. Let $f : \left((\mathbb{R}^m)^2\right)^k \longrightarrow \mathbb{R}$ be any measurable function such that $f\left(\underline{\xi,-\xi}_k\right)$ is $\mathcal{S}^1\left(\underline{\xi,-\xi}_k\right)$-measurable. This means $f\left(\pi\left(e\left(\underline{\xi,-\xi}_k\right)\right)\right) = f\left(\underline{\xi,-\xi}_k\right)$ for all $\pi\in\Pi_k$, and all $e\in\Xi_k$.

To complete this proof we have to prove that $f\left(\underline{\xi,-\xi}_k\right)$ is $\mathcal{S}^1\left(\underline{\eta,-\eta}_k,\mathcal{C}\right)$-measurable. For this, let $\underline{z}_k = (z_1,\cdots,z_k) \in (\mathbb{R}^m)^k$, $B\in\mathcal{M}_{m\times m}$, and B is positive definite and symmetric, h be defined by

$$h\left(\underline{z,-z}_k, B\right) = \left(\left(B^{\frac{1}{2}}z_1, -B^{\frac{1}{2}}z_1\right), \cdots, \left(B^{\frac{1}{2}}z_k, -B^{\frac{1}{2}}z_k\right)\right)^t.$$

It is easy to see that $h\left(\pi(e(z,-z)), B\right) = \pi\left(e\left(h\left(\underline{z,-z}_k, B\right)\right)\right)$, $\underline{\xi,-\xi}_k = h\left(\underline{\eta,-\eta}_k, V(\underline{\xi}_k)\right)$, and $f\left(\underline{\xi,-\xi}_k\right) = f\left(h\left(\underline{\eta,-\eta}_k, V(\underline{\xi}_k)\right)\right)$ but $V\left(\underline{\xi}_k\right)$ is $\mathcal{S}^1\left(\underline{\eta,-\eta}_k, \mathcal{C}\right)$-measurable because $\mathcal{C} \subseteq \mathcal{S}^1\left(\underline{\eta,-\eta}_k, \mathcal{C}\right)$, and it is $\mathcal{C}$-measurable. But also, $f\left(h\left(\pi\left(e\left(\underline{\eta,-\eta}_k\right)\right), V(\underline{\xi}_k)\right)\right) = f\left(h\left(\underline{\eta,-\eta}_k, V(\underline{\xi}_k)\right)\right)$ for all $\pi \in \Pi_k$, and all $e \in \Xi_k$.
Thus, $f\left(\underline{\xi,-\xi}_k\right) = f\left(h\left(\underline{\eta,-\eta}_k, V(\underline{\xi}_k)\right)\right)$ is $\mathcal{S}^1\left(\underline{\eta,-\eta}_k, \mathcal{C}\right)$-measurable. □

In the introduction of this section we have defined the statistic
$T_n := \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \omega_{ni} f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right)$, we want here to make some preliminary computations to prepare for the next theorem which will discuss the conditional asymptotic normality of the sequence $(T_n)_{n\in\mathbb{N}}$. For this, let us first prove that $E\left(f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right) | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)$, and
$Var\left(f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right) | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)$, are independent of i. One can easily see

$$E\left(f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right) | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right) = 0 \ [\mathbb{P}],$$

$Var\left(f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right) | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right) =$
$= \frac{1}{k_n} \sum_{i=1}^{k_n} f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right) \cdot \left(f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right)\right)^t$ $[\mathbb{P}]$. Therefore, for notational convenience we denote
$E\left(f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right) | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)$ by $E\left(f_n | \mathcal{S}^1\left(\underline{X}_{k_n}\right)\right)$, i.e.

$$E\left(f_n | \mathcal{S}^1\left(\underline{X}_{k_n}\right)\right) := E\left(f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right) | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right),$$

and $Var\left(f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right) | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)$ by $Var\left(f_n | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)$, i.e.

$$Var\left(f_n | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right) := Var\left(f_n\left((X_{ni}, Y_{ni}), \underline{X,Y}_{k_n}\right) | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right).$$

Let us now give explicit expressions for $E\left(T_n | \mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)$, and

$Var\left(T_n|\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right).$

$$E\left(T_n|\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)=\frac{1}{\sqrt{k_n}}\sum_{i=1}^{k_n}\omega_{ni}E\left(f_n|\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)=0\ [\mathbb{P}],$$

$$\begin{aligned}
&Var\left(T_n|\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)=\\
&=Var\left(\tfrac{1}{\sqrt{k_n}}\sum_{i=1}^{k_n}\omega_{ni}f_n\left(\left(X_{ni},Y_{ni}\right),\underline{X,Y}_{k_n}\right)|\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)\ [\mathbb{P}]\\
&=\tfrac{1}{k_n}\sum_{i=1}^{k_n}\omega_{ni}Var\left(f_n|\mathcal{S}^0\left(\underline{X}_{k_n}\right)\right)\omega_{ni}^t+\\
&+\tfrac{1}{k_n}\sum_{1\leqslant i\neq j\leqslant k_n}\omega_{ni}E\Bigg(f_n\left(\left(X_{ni},Y_{ni}\right),\underline{X,Y}_{k_n}\right)\cdot\\
&\qquad\cdot\left(f_n\left(\left(X_{ni},Y_{ni}\right),\underline{X,Y}_{k_n}\right)\right)^t|\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\Bigg)\omega_{nj}^t\ [\mathbb{P}].
\end{aligned}$$

Hence,

$Var\left(T_n|\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)=\frac{1}{k_n}\sum_{i=1}^{k_n}\omega_{ni}Var\left(f_n|\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)\omega_{ni}^t\ [\mathbb{P}]$

– For the next theorem we assume first the same situation in the introduction of this section.

Theorem 3.2.15. Assume that the triangular array

$$\frac{1}{\sqrt{k_n}}f_n\left(\left(X_{ni},Y_{ni}\right),\underline{X,Y}_{k_n}\right),\ 1\leqslant i\leqslant k_n,\ n\in\mathbb{N},$$

is infinitesimal and the sequence $\left(Var\left(f_n|\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)\right)_{n\in\mathbb{N}}$ is stochastically bounded. Assume further that the sequence $\left(\omega_n(t)\right)_{n\in\mathbb{N}}\subseteq\mathcal{L}^2\left([0,1],\mathcal{M}_{m\times d}\right)$ is relatively compact sequence of non-random weight functions. Then the sequence $\left(T_n\right)_{n\in\mathbb{N}}$ is asymptotically normal under $\mathbb{P}\in H_1$, and conditioned by the symmetric sub-σ-field $\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)$. More explicitly,

$$\mathbb{P}*T_n\sim\mathcal{N}\left(0,Var\left(T_n|\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right)\right)\ \left(\mathcal{S}^1\left(\underline{X,Y}_{k_n}\right)\right).$$

Proof: Let ξ_{ni}, $i=1,2,\cdots,k_n$, $n\in\mathbb{N}$, be defined by

$$\xi_{ni}:=\frac{1}{\sqrt{k_n}}f_n\left(\left(X_{ni},Y_{ni}\right),\underline{X,Y}_{k_n}\right),\ 1\leqslant i\leqslant k_n,\ n\in\mathbb{N}.$$

From lemma 3.2.13, the random arrays $\xi_{n1}, \cdots, \xi_{nk_n}$, are H_1-exchangeable under $\mathbb{P}\left(\cdot, \mathcal{S}^1\left(\underline{X, Y}_{k_n}\right)\right)$. Let

$$F_n(t) := \sum_{i=1}^{[k_n t]} \frac{1}{\sqrt{k_n}} f_n\left(\left(X_{ni}, Y_{ni}\right), \underline{X, Y}_{k_n}\right),$$

$0 \leqslant t \leqslant 1$, and let

$$F_n'(t) := F_n\left(\frac{i-1}{k_n}\right) + k_n\left(t - \frac{i-1}{k_n}\right)\left(F_n\left(\frac{i}{k_n}\right) - F_n\left(\frac{i-1}{k_n}\right)\right),$$

where $i := [k_n t] + 1$.
Therefore, we can write

$$T_n = \int_0^1 \omega_n(t) dF_n'(t).$$

Hence, all assumption of theorem 3.2.9 are satisfied, and consequently we have

$$\mathbb{P} * T_n \sim \mathcal{N}\left(0, \sigma_n^2\right) \ \left(\mathcal{S}^1\left(\underline{X, Y}_{k_n}\right)\right).$$

where $\sigma_n^2 := \int_0^1 \omega_n(t) s^2\left(\underline{\xi}_{k_n}\right) \omega_n^t(t) dt =$
$= \int_0^1 \omega_n(t) Var\left(f_n | \mathcal{S}^1\left(\underline{X, Y}_{k_n}\right)\right) \omega_n^t(t) dt$. Also, from remark 3.2.10 we can write
$\sigma_n^2 = \frac{1}{k_n} \sum_{i=1}^{k_n} \omega_{ni} s^2(\underline{\xi}_{k_n}) \omega_{ni}^t$
$\sigma_n^2 = \frac{1}{k_n} \sum_{i=1}^{k_n} \omega_{ni} Var\left(f_n | \mathcal{S}^1\left(\underline{X, Y}_{k_n}\right)\right) \omega_{ni}^t \ [\mathbb{P}]$.
By remarking that $\sigma_n^2 - Var\left(T_n | \mathcal{S}^1\left(\underline{X, Y}_{k_n}\right)\right) \xrightarrow{\mathbb{P}} 0$, we conclude that the assertion is valid. $\square$

3.3. The Hypothesis H_2 (independence):

Introduction: Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathbb{P}$ be a probability measure defined on $\mathcal{A}$, $\underline{X,Y}_{k_n} = ((X_{n1}, Y_{n1}),\ (X_{n2}, Y_{n2}),\ \cdots,\ (X_{nk_n}, Y_{nk_n}))^t$, $n \in \mathbb{N}$, be a triangular array of random couples with values in a sample space $(\mathcal{E}^2, \mathcal{B}^2)$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. Assume that the random couples of each row (i.e. $(X_{n1}, Y_{n1}),\ \cdots,\ (X_{nk_n}, Y_{nk_n})$, $n \in \mathbb{N}$) are i.i.d. under $\mathbb{P}$, and the random elements of each couple are also independent under $\mathbb{P}$. Let the distribution of (X_{ni}, Y_{ni}) under $\mathbb{P}$ be denoted by $\mathbb{P} * (X_{ni}, Y_{ni})$ or shortly by P_n. The general form of linear statistics which are considered here is denoted by $T_n\left(\underline{X,Y}_{k_n}\right)$, or shortly by T_n if we can avoid confusion, where $T_n := \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \omega_{ni} f_n\left(X_{ni}, \underline{X}_{k_n}\right)\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right)\right)^t$, and where $\omega_{ni} \in \mathcal{M}_{m \times d_1}$, $i = 1, \cdots, k_n$, $n \in \mathbb{N}$, where the matrices ω_{ni}, $i = 1, \cdots, k_n$, $n \in \mathbb{N}$, are assumed to be given constants. And we want here to introduce a new concept of symmetry
w depends on $\underline{x,y}_{k_n}$ in a permutation symmetric way iff

$$w\left(\lambda\left(\underline{x,y}_{k_n}\right)\right) = w\left(\underline{x,y}_{k_n}\right), \forall\ \lambda \in \Lambda_{k_n}.$$

We mention here that this is the symmetry concept, which is used in all arguments related with this section, and the symbols λ, Λ_{k_n} are introduced just before definition 3.3.1. The functions $f_n : \mathcal{E} \times \mathcal{E}^{k_n} \longrightarrow \mathbb{R}^{d_1}$, $(x, \underline{z}_{k_n}) \longmapsto f_n(x, \underline{z}_{k_n})$, and $g_n : \mathcal{E} \times \mathcal{E}^{k_n} \longrightarrow \mathbb{R}^{d_2}$, $(y, \underline{z}_{k_n}) \longmapsto g_n(y, \underline{z}_{k_n})$, $n \in \mathbb{N}$, are measurable and such that they depend on $\underline{z}_{k_n}$ in a permutation symmetric way. Fix $n \in \mathbb{N}$, and let $\lambda_1 \in \mathbb{R}^{d_1}$ be a given constant, define $\Omega_{\lambda_1} \subseteq \Omega$ by

$$\Omega_{\lambda_1} := \left\{\lambda_1^t f_n\left(X_{n1}, \underline{X}_{k_n}\right) = \lambda_1^t f_n\left(X_{n2}, \underline{X}_{k_n}\right) = \cdots = \lambda_1^t f_n\left(X_{nk_n}, \underline{X}_{k_n}\right)\right\}$$

we assume that $\mathbb{P}(\Omega_{\lambda_1}) > 0$ is hold iff $\lambda_1 = 0$. Also similarly, let $\lambda_2 \in \mathbb{R}^{d_2}$ be a given constant, and define $\Omega_{\lambda_2} \subseteq \Omega$ by

$$\Omega_{\lambda_2} := \left\{\lambda_2^t g_n\left(Y_{n1}, \underline{Y}_{k_n}\right) = \lambda_2^t g_n\left(Y_{n2}, \underline{Y}_{k_n}\right) = \cdots = \lambda_2^t g_n\left(Y_{nk_n}, \underline{Y}_{k_n}\right)\right\}$$

we assume further that $\mathbb{P}(\Omega_{\lambda_2}) > 0$ is hold iff $\lambda_2 = 0$. Now for notational convenience concerning the computations, let $\omega_n \in \mathcal{L}^2([0,1], \mathcal{M}_{m \times d_1})$ be such

that $\omega_{ni} = k_n \int\limits_{\frac{i-1}{k_n}}^{\frac{i}{k_n}} \omega_n(t)dt$, $i = 1, \cdots, k_n$, $n \in \mathbb{N}$.
We mention here that the form of the considered statistic T_n contains the linear rank test statistic $S = \sum\limits_{i=1}^{n} a(R_i)b(Q_i)$ which has been discussed deeply in "Theory of rank tests [1967]" which is due to J. Hájek and Z. Šidák.

To achieve the purpose here we need to build a base for our coming limit theorems, for this we shall begin with the following interesting steps which are needed very much to obtain the final results.

Let us here introduce another new kind of exchangeablity, we call it H_2-exchangeablity, also we put some related symbols, which will be needed for the main results of this section.
Let $(\Omega, \mathcal{A}, P)$ be a probability space and let

$$\underline{\xi}_{k_n} = (\xi_{n1}, \cdots, \xi_{nk_n})^t,\ \underline{\tilde{\xi}}_{k_n} = (\tilde{\xi}_{n1}, \cdots, \tilde{\xi}_{nk_n})^t$$

be triangular arrays of random elements from $(\Omega, \mathcal{A}, P)$ to $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. let us define $\underline{\xi, \tilde{\xi}}_{k_n} := \left((\xi_{n1}, \tilde{\xi}_{n1}), \cdots, (\xi_{nk_n}, \tilde{\xi}_{nk_n})\right)$, and let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence of sub-σ-fields of $\mathcal{A}$, $\mathcal{C}$ be a sub-σ-field of $\mathcal{A}$. Let further $\mathcal{S}^2\left(\underline{\xi, \tilde{\xi}}_{k_n}\right) := \left(\underline{\xi, \tilde{\xi}}_{k_n}\right)^{-1}(\mathcal{S}_n^2)$, where $\mathcal{S}_n^2$ denotes here the σ-field of all sets B in $(\mathcal{B}^2)^{k_n}$ satisfying the condition

$$\underline{x, \tilde{x}}_{k_n} \in B \Longleftrightarrow \lambda\left(\underline{x, \tilde{x}}_{k_n}\right) \in B,\ \forall \lambda \in \Lambda_{k_n},$$

where here $\underline{x, \tilde{x}}_{k_n} := ((x_1, \tilde{x}_1), \cdots, (x_{k_n}, \tilde{x}_{k_n}))$, and $\Lambda_{k_n} := \Big\{\lambda : \lambda\left(\underline{x, \tilde{x}}_{k_n}\right) = \left(\left(x_{\pi_1(1)}, \tilde{x}_{\pi_2(1)}\right), \cdots, \left(x_{\pi_1(k_n)}, \tilde{x}_{\pi_2(k_n)}\right)\right)$, for some π_1, π_2 of the bijective functions defined on $\{1, 2, \cdots, k_n\}$, $\forall\ \underline{x, \tilde{x}}_{k_n} \in (\mathcal{E}^2)^{k_n}\Big\}$.
Let $s^2(\underline{\xi}_{k_n}) := \sum\limits_{i=1}^{k_n} \left(\xi_{ni} - \bar{\xi}_n\right)\left(\xi_{ni} - \bar{\xi}_n\right)^t$, and $\bar{\xi}_n := \frac{1}{k_n}\sum\limits_{i=1}^{k_n} \xi_{ni}$. For each $n \in \mathbb{N}$ the matrix $s^2(\underline{\xi}_{k_n})$ is assumed to be positive definite, also we impose the same conditions on the matrices $s^2(\underline{\eta}_{k_n})$. All these assumptions and conditions are hold in this section.

Definition 3.3.1. The random pairs $(\xi_{n1}, \tilde{\xi}_{n1}), \cdots, (\xi_{nk_n}, \tilde{\xi}_{nk_n})$ are H_2-exchangeable under $P(\cdot|\mathcal{C}_n)$, iff $\forall \lambda \in \Lambda_{k_n}$ the equality

$$P\left(\underline{\xi, \tilde{\xi}}_{k_n} \in B|\mathcal{C}_n\right) = P\left(\lambda\left(\underline{\xi, \tilde{\xi}}_{k_n}\right) \in B|\mathcal{C}_n\right)\ [P]$$

is valid for all $B \in (\mathcal{B}^2)^{k_n}$, and then the array $\underline{\xi, \tilde{\xi}}_{k_n}$ is called H_2-exchangeable under $P(\cdot|\mathcal{C}_n)$.

Lemma 3.3.2. Let $f : (\mathcal{E}^2, \mathcal{B}^2)^{k_n} \longrightarrow (\mathbb{R}, \mathbb{B})$ be a measurable function such that $E(f(\underline{\xi, \tilde{\xi}}_{k_n}))$ is well-defined. If $\underline{\xi, \tilde{\xi}}_{k_n}$ is H_2-exchangeable under $P(\cdot|\mathcal{C})$, then $\forall A \in \mathcal{S}^2(\underline{\xi, \tilde{\xi}}_{k_n}),\ C \in \mathcal{C},\ \lambda \in \Lambda_{k_n}$ the following equality is valid

$$E\left(1_{A\cap C} f\left(\lambda\left(\underline{\xi, \tilde{\xi}}_{k_n}\right)\right)\right) = E\left(1_{A\cap C} f\left(\underline{\xi, \tilde{\xi}}_{k_n}\right)\right).$$

□

Let us denote the σ-field $\sigma(\mathcal{S}^2\left(\underline{\xi, \tilde{\xi}}_{k_n}\right), \mathcal{C})$ by $\mathcal{S}^2(\underline{\xi, \tilde{\xi}}_{k_n}, \mathcal{C})$. Now, since $\{A \cap C : A \in \mathcal{S}^2\left(\underline{\xi, \tilde{\xi}}_{k_n}\right),\ C \in \mathcal{C}\}$ generates the σ-field $\mathcal{S}^2(\underline{\xi, \tilde{\xi}}_{k_n}, \mathcal{C})$, we have by lemma 3.3.2 the validity of

$$E\left(1_D f\left(\lambda\left(\underline{\xi, \tilde{\xi}}_{k_n}\right)\right)\right) = E\left(1_D f\left(\underline{\xi, \tilde{\xi}}_{k_n}\right)\right),$$

$\forall D \in \mathcal{S}^2(\underline{\xi, \tilde{\xi}}_{k_n}, \mathcal{C}),\ \lambda \in \Lambda_{k_n}$.
Therefore, the following lemma is just a consequence of lemma 3.3.2.

Lemma 3.3.3. Let $f : (\mathcal{E}^2, \mathcal{B}^2)^{k_n} \longrightarrow (\mathbb{R}, \mathbb{B})$ be a measurable function such that $E(f(\underline{\xi, \tilde{\xi}}_{k_n}))$ is well-defined. If $\underline{\xi, \tilde{\xi}}_{k_n}$ is H_2-exchangeable under $P(\cdot|\mathcal{C})$, then

$$E(f(\underline{\xi, \tilde{\xi}}_{k_n})|\mathcal{S}^2\left(\underline{\xi, \tilde{\xi}}_{k_n}, \mathcal{C}\right)) = \frac{1}{(k_n!)^2} \sum_{\lambda\in\Lambda_{k_n}} f(\lambda(\underline{\xi, \tilde{\xi}}_{k_n}))\ [P].$$

Let us now introduce the following invariance principle which is needed to build the proofs of the coming limit theorems of this section.

Theorem 3.3.4. Suppose that the triangular arrays $\underline{\xi}_{k_n}$, $\underline{\tilde{\xi}}_{k_n}$ satisfying the following conditions:

The sequences $\left(\frac{1}{\sqrt{k_n}} s^2(\underline{\xi}_{k_n}\right)_{n\in\mathbb{N}}$, $\left(\frac{1}{\sqrt{k_n}} s^2(\underline{\tilde{\xi}}_{k_n})\right)_{n\in\mathbb{N}}$, $\left(\frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \xi_{ni}\right)_{n\in\mathbb{N}}$, and $\left(\frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \tilde{\xi}_{ni}\right)_{n\in\mathbb{N}}$ are stochastically bounded, where

$s^2(\underline{\xi}_{k_n}) := \sum_{i=1}^{k_n} (\xi_{ni} - \bar{\xi}_n)(\xi_{ni} - \bar{\xi}_n)^t$,

and

$s^2(\underline{\tilde{\xi}}_{k_n}) := \sum_{i=1}^{k_n} (\tilde{\xi}_{ni} - \bar{\tilde{\xi}}_n)(\tilde{\xi}_{ni} - \bar{\tilde{\xi}}_n)^t$,

and where $\bar{\xi}_n := \frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{ni}$, and $\bar{\tilde{\xi}}_n := \frac{1}{k_n} \sum_{i=1}^{k_n} \tilde{\xi}_{ni}$. And we suppose further that the arrays $\underline{\xi}_{k_n}$, and $\underline{\tilde{\xi}}_{k_n}$ are infinitesimal. If $\mathcal{C}_n \supseteq \mathcal{S}^2\left(\underline{\xi, \tilde{\xi}}_{k_n}\right)$, $n \in \mathbb{N}$, is a sequence of sub-σ-fields of $\mathcal{A}$, $\tilde{\mathcal{C}}_n := \sigma(C \times (C[0,1])^{d_1\times d_2} : C \in \mathcal{C}_n)$, and if the triangular array $\underline{\xi, \tilde{\xi}}_{k_n}$ is H_2-exchangeable under $P(\cdot|\mathcal{C}_n)$ then

$$\tilde{\mathcal{S}}_n \overset{w}{\sim} \tilde{W}\ (\tilde{\mathcal{C}}_n).$$

Where

$$\tilde{\mathcal{S}}_n : \Omega \times (C[0,1])^{d_1\times d_2} \longrightarrow (D[0,1])^{d_1\times d_2},$$
$$\tilde{\mathcal{S}}_n(\omega_1, \omega_2) := \mathcal{S}_n(\omega_1),$$
$$(\mathcal{S}_n(\omega_1))(t) := \sum_{i=1}^{[k_n t]} (\xi_{ni}\tilde{\xi}_{ni}^t)(\omega_1).$$

And

$$\tilde{W} : \Omega \times (C[0,1])^{d_1\times d_2} \longrightarrow (C[0,1])^{d_1\times d_2} \subset (D[0,1])^{d_1\times d_2},$$

$\tilde{W}_t(\omega_1, \omega_2) := \frac{1}{\sqrt{k_n}} \left(\sqrt{s^2(\underline{\xi}_{k_n})}\right)(\omega_1) W_t(\omega_2) \left(\sqrt{s^2(\underline{\tilde{\xi}}_{k_n})}\right)(\omega_1) +$

$$+ \frac{t}{k_n} \sum_{i=1}^{k_n} \xi_{ni}(\omega_1) \sum_{i=1}^{k_n} \tilde{\xi}_{ni}^t(\omega_1),\ \forall t \in [0,1],\ \forall (\omega_1, \omega_2) \in \Omega \times (C[0,1])^{d_1\times d_2},$$

where $W := (W_t)_{0\leqslant t\leqslant 1} = \left(\left(W_t^{jk}\right)_{0\leqslant t\leqslant 1}\right)_{\substack{1\leqslant j\leqslant d_1 \\ 1\leqslant k\leqslant d_2}}$ is a $d_1 \times d_2$-dimensional Brownian motion.

Proof: We prove it first in the case where $\sup_n \| \frac{1}{\sqrt{k_n}} s^2(\underline{\xi}_{k_n}) \|$, $\sup_n \| \frac{1}{\sqrt{k_n}} s^2(\underline{\tilde{\xi}}_{k_n}) \|$, $\sup_n \| \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \xi_{ni} \|$, and $\sup_n \| \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \tilde{\xi}_{ni} \|$ are strictly less

than infinity almost everywhere with respect to P.
By lemma 1.2.2 the assertion is equivalent to the validity of the limit

$$\int\limits_{C_n\times(C[0,1])^{d_1\times d_2}} \varphi(\tilde{\mathcal{S}}_n)dP\otimes\mu - \int\limits_{C_n\times(C[0,1])^{d_1\times d_2}} \varphi(\tilde{W})dP\otimes\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

for all sequences $(C_n)_{n\in\mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi:(D[0,1])^{d_1\times d_2}\longrightarrow\mathbb{R}$, where $C_n\in\mathcal{C}_n$, and where μ is the Wiener measure defined on $(C[0,1])^{d_1\times d_2}$.
Therefore, let $(C_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of the measurable sets $C_n\in\mathcal{C}_n$, $n\in\mathbb{N}$, we want to prove the validity of the previous limit above, which is equivalent to

$$\int\limits_{C_n}\varphi(\mathcal{S}_n)dP - \int\limits_{C_n}\int\limits_{(C[0,1])^{d_1\times d_2}}\varphi(\tilde{W})dPd\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0$$

$$\Longleftrightarrow \int\limits_{C_n} E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right)dP - \int\limits_{C_n} E\left(\int\limits_{(C[0,1])^{d_1\times d_2}}\varphi(\tilde{W})d\mu\middle|\mathcal{C}_n\right)dP \underset{n\longrightarrow\infty}{\longrightarrow} 0$$

Since the sequence $(C_n)_{n\in\mathbb{N}}$ is arbitrary, the previous assertion is equivalent to

$$E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right) - E\left(\int\limits_{(C[0,1])^{d_1\times d_2}}\varphi(\tilde{W})d\mu\middle|\mathcal{C}_n\right)\overset{P}{\longrightarrow}0$$

$$\Longleftrightarrow E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right) - \int\limits_{(C[0,1])^{d_1\times d_2}}\varphi(\tilde{W})d\mu\overset{P}{\longrightarrow}0$$

$$\Longleftrightarrow \frac{1}{(k_n!)^2}\sum_{\pi_1,\pi_2}\varphi\left(\sum_{i=1}^{[k_n(\cdot)]}\xi_{n\pi_1(i)}\tilde{\xi}^t_{n\pi_2(i)}\right) - \int\limits_{(C[0,1])^{d_1\times d_2}}\varphi(\tilde{W})d\mu\overset{P}{\longrightarrow}0,$$

where π_1,π_2 here ranging over all bijective functions, defined on $\{1,2,\cdots,k_n\}$.
It is sufficient to prove that each subsequence $\{n'\}$ contains another sub-subsequence $\{n''\}$ such that

$$\Longleftrightarrow \frac{1}{(k_{n''}!)^2}\sum_{\pi_1,\pi_2}\varphi\left(\sum_{i=1}^{[k_{n''}(\cdot)]}\xi_{n''\pi_1(i)}\tilde{\xi}^t_{n''\pi_2(i)}\right) - \int\limits_{(C[0,1])^{d_1\times d_2}}\varphi(\tilde{W})d\mu\overset{P}{\longrightarrow}0\ [P].$$

From the hypotheses, each subsequence $\{n'\}$ of $\{n\}$, contains another sub-subsequence $\{n''\}$ such that

$$\max_{1\leqslant i\leqslant k_{n''}} \| \xi_{n''i} \| \longrightarrow 0\ [P],$$

and

$$\max_{1\leqslant i\leqslant k_{n''}} \| \tilde{\xi}_{n''i} \| \longrightarrow 0\ [P].$$

Therefore, without any loss of generality we can assume that

$$\max_{1\leqslant i\leqslant k_n} \| \xi_{ni} \| \longrightarrow 0\ [P],$$

and

$$\max_{1\leqslant i\leqslant k_n} \| \tilde{\xi}_{ni} \| \longrightarrow 0\ [P].$$

Hence, for all ω_1 of those fulfill the hypotheses, we want now to prove

$$\frac{1}{(k_n!)^2}\sum_{\pi_1,\pi_2}\varphi\left(\sum_{i=1}^{[k_n(\cdot)]}\xi_{n\pi_1(i)}(\omega_1)\tilde{\xi}^t_{n\pi_2(i)}(\omega_1)\right)-$$

$$-\int_{(C[0,1])^{d_1\times d_2}}\varphi\Bigg\{\frac{1}{\sqrt{k_n}}\left(\sqrt{s^2(\underline{\xi}_{k_n})}\right)(\omega_1)W\left(\sqrt{s^2(\underline{\tilde{\xi}}_{k_n})}\right)(\omega_1)+$$

$$+\frac{(\cdot)}{k_n}\sum_{i=1}^{k_n}\xi_{ni}(\omega_1)\sum_{i=1}^{k_n}\tilde{\xi}^t_{ni}(\omega_1)\Bigg\}d\mu\longrightarrow 0.$$

But this is valid indeed from remark 2.3.8.
Now, we turn to prove it in the general case.
Let us define $C_n^M := \left\{ \| \frac{1}{\sqrt{k_n}}\sum_{i=1}^{k_n}\xi_{ni} \| \leqslant M,\ \| \frac{1}{\sqrt{k_n}}s^2(\underline{\xi}_{k_n}) \| \leqslant M\right\}$, and $\tilde{C}_n^M := \left\{ \| \frac{1}{\sqrt{k_n}}\sum_{i=1}^{k_n}\tilde{\xi}_{ni} \| \leqslant M,\ \| \frac{1}{\sqrt{k_n}}s^2(\underline{\tilde{\xi}}_{k_n}) \| \leqslant M\right\}$ and also for each $n\in\mathbb{N}$, and $i=1,\cdots,k_n$, we define the new random arrays $\eta_{ni} := 1_{C_n^M}\cdot\xi_{ni}$, $\tilde{\eta}_{ni} := 1_{\tilde{C}_n^M}\cdot\tilde{\xi}_{ni}$. It is clear that the new random arrays satisfy the first case of the proof above.
Consequently, the limit

$$\int_{C_n\times(C[0,1])^{d_1\times d_2}}\varphi(1_{C_n^M\cap\tilde{C}_n^M}\cdot\tilde{\mathcal{S}}_n)dP\otimes\mu-\int_{C_n\times(C[0,1])^{d_1\times d_2}}\varphi(1_{C_n^M\cap\tilde{C}_n^M}\cdot\tilde{W})dP\otimes\mu\underset{n\longrightarrow\infty}{\longrightarrow}0,$$

is valid for all sequences $(C_n)_{n\in\mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi:(D[0,1])^{d_1d_2}\longrightarrow\mathbb{R}$, where $C_n\in\mathcal{C}_n$. We can rewrite the previous limit as the following

$$\int_{C_n\times(C[0,1])^{d_1\times d_2}}1_{C_n^M\cap\tilde{C}_n^M}\cdot\varphi(\tilde{\mathcal{S}}_n)dP\otimes\mu-\int_{C_n\times(C[0,1])^{d_1\times d_2}}1_{C_n^M\cap\tilde{C}_n^M}\cdot\varphi(\tilde{W})dP\otimes\mu\underset{n\longrightarrow\infty}{\longrightarrow}0.$$

Also, we can enlarge M to make the following inequality valid $P(C_n^M \cap \tilde{C}_n^M) \geqslant 1 - \varepsilon$ for any given $\varepsilon > 0$, and for all $n \in \mathbb{N}$.
Therefore, we obtain the validity of the limit

$$\int\limits_{C_n \times (C[0,1])^{d_1 \times d_2}} \varphi(\tilde{\mathcal{S}}_n) dP \otimes \mu - \int\limits_{C_n \times (C[0,1])^{d_1 \times d_2}} \varphi(\tilde{W}) dP \otimes \mu \underset{n \longrightarrow \infty}{\longrightarrow} 0,$$

for all sequences $(C_n)_{n \in \mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi : (D[0,1])^{d_1 \times d_2} \longrightarrow \mathbb{R}$, where $C_n \in \mathcal{C}_n$.
Hence, the proof is complete. $\square$

Corollary 3.3.5. In the same situation of theorem 3.3.4, we have the validity of

$$\tilde{\mathcal{S}}_n \overset{w}{\sim} \tilde{W},$$

or more explicitly

$$\left(\sum_{i=1}^{[k_n t]} \xi_{ni} \tilde{\xi}_{ni}^t \right)_{0 \leqslant t \leqslant 1} \overset{w}{\sim} \left(\frac{1}{\sqrt{k_n}} \sqrt{s^2(\underline{\xi}_{k_n})}\ W_t \sqrt{s^2(\underline{\tilde{\xi}}_{k_n})} + \frac{t}{k_n} \sum_{i=1}^{k_n} \xi_{ni} \sum_{i=1}^{k_n} \tilde{\xi}_{ni}^t \right)_{0 \leqslant t \leqslant 1}.$$

In the same situation of theorem 3.3.4, we want to discuss the conditional asymptotic normality of a given linear combination of the random arrays $S_n(t_1)$, $S_n(t_2)$,$\cdots$, $S_n(t_k)$, where t_1, t_2,$\cdots$, t_k belong to $[0,1]$. The following remark is for this purpose.

Remark 3.3.6. Let X_n be defined by $\begin{pmatrix} S_n(t_1) \\ \vdots \\ S_n(t_k) \end{pmatrix}$, where t_1, t_2,$\cdots$, t_k belong to $[0,1]$, and let a_1, a_2,$\cdots$, a_k be given constant matrices of order $m \times d_1$, then we have
$P * \sum\limits_{i=1}^{k} a_i \cdot S_n(t_i) \sim \mathcal{N}(\mu_n, \sigma_n^2)\ \ (\mathcal{C}_n)$, where $\forall n \in \mathbb{N}$

$$\mu_n := \sum_{j=1}^{k} t_j a_j \cdot \left(\frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{ni} \sum_{i=1}^{k_n} \eta_{ni}^t \right),$$

$$\sigma_n^2 := (a_1, \cdots, a_k) \overset{\rightarrow}{\bullet} \left(\left(\min(t_i, t_j) \frac{1}{k_n} s^2\left(\underline{\xi}_{k_n}\right) \right)_{1 \leqslant i,j \leqslant k} \overset{\rightarrow}{\star} s^2\left(\underline{\eta}_{k_n}\right) \right) \overset{\leftarrow}{\bullet} \begin{pmatrix} a_1^t \\ \vdots \\ a_k^t \end{pmatrix}.$$

Proof: From Theorem 3.3.4 we conclude the following fact

$$P * \begin{pmatrix} S_n(t_1) \\ \vdots \\ S_n(t_k) \end{pmatrix} \sim \mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right) \ (\mathcal{C}_n),$$

where here $\forall n \in \mathbb{N}$

$$\tilde{\mu}_n := \begin{pmatrix} t_1 \frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{ni} \sum_{i=1}^{k_n} \eta_{ni}^t \\ \vdots \\ t_k \frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{ni} \sum_{i=1}^{k_n} \eta_{ni}^t \end{pmatrix},$$

$$\tilde{\sigma}_n^2 := \left(\min(t_i, t_j) \frac{1}{k_n} s^2(\underline{\xi}_{k_n})\right)_{1 \leqslant i,j \leqslant k} \vec{\star}\ s^2(\underline{\eta}_{k_n}).$$

Which means by definition the validity of the following limit
$E_P\left(f\left(S_n(t_1), S_n(t_2), \cdots, S_n(t_k)\right) | \mathcal{C}_n\right) - \int f d\mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right) \xrightarrow{P} 0$, for all bounded uniformly continuous functions $f : (\mathcal{M}_{d_1 \times d_2})^k \longrightarrow \mathbb{R}$. Now, let $g : \mathcal{M}_{m \times d_2} \longrightarrow \mathbb{R}$ be a bounded uniformly continuous function, and let $a_1,\ a_2, \cdots,\ a_k \in \mathcal{M}_{m \times d_1}$ be given constant matrices, then the function $h : (\mathcal{M}_{d_1 \times d_2})^k \longrightarrow \mathbb{R}$, defined by $h(x) := g((a_1, a_2, \cdots, a_k) \cdot x)$, $\forall x \in (\mathcal{M}_{d_1 \times d_2})^k$ is also bounded uniformly continuous function. Thus, we have for this h the validity of the limit
$E_P\left(h\left(S_n(t_1), S_n(t_2), \cdots, S_n(t_k)\right) | \mathcal{C}_n\right) - \int h d\mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right) \xrightarrow{P} 0$, which can be rewritten as
$E_P\left(g\left((a_1, a_2, \cdots, a_k) \cdot \begin{pmatrix} S_n(t_1) \\ \vdots \\ S_n(t_k) \end{pmatrix}\right) \Bigg| \mathcal{C}_n\right) - \int g d\mathcal{N}\left(\mu_n, \sigma_n^2\right) \xrightarrow{P} 0$, which is also equivalent to

$$E_P\left(g\left(\sum_{j=1}^{k} a_j \cdot S_n(t_j)\right) | \mathcal{C}_n\right) - \int g d\mathcal{N}\left(\mu_n, \sigma_n^2\right) \xrightarrow{P} 0,$$

where here

$$\mu_n := (a_1, a_2, \cdots, a_k) \cdot \begin{pmatrix} t_1 \frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{ni} \sum_{i=1}^{k_n} \eta_{ni}^t \\ \vdots \\ t_k \frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{ni} \sum_{i=1}^{k_n} \eta_{ni}^t \end{pmatrix} = \sum_{j=1}^{k} t_j a_j \cdot \left(\frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{ni} \sum_{i=1}^{k_n} \eta_{ni}^t\right),$$

and

$$\sigma_n^2 := (a_1, \cdots, a_k) \overrightarrow{\bullet} \left(\left(\min(t_i, t_j) \frac{1}{k_n} s^2 \left(\underline{\xi}_{k_n} \right) \right)_{1 \leqslant i,j \leqslant k} \overrightarrow{\star} s^2 \left(\underline{\eta}_{k_n} \right) \right) \overleftarrow{\bullet} \begin{pmatrix} a_1^t \\ \vdots \\ a_k^t \end{pmatrix}.$$

Therefore, the proof is complete. □

Corollary 3.3.7. In remark 3.3.6 if $0 < t_1 < t_2 < \cdots < t_k < 1$, then we can rewrite σ_n^2 as the following

$$\sigma_n^2 = \sum_{j=1}^{k} \left(\sum_{i=1}^{j} a_i \overrightarrow{\bullet} \left(\tfrac{1}{k_n} s^2(\underline{\xi}_{k_n}) \overrightarrow{\star} s^2(\underline{\eta}_{k_n}) \right) \overleftarrow{\bullet} a_j^t t_i + \right.$$
$$\left. + \sum_{i=1}^{k-j} a_{j+i} \overrightarrow{\bullet} \left(\tfrac{1}{k_n} s^2(\underline{\xi}_{k_n}) \overrightarrow{\star} s^2(\underline{\eta}_{k_n}) \right) \overleftarrow{\bullet} a_j^t t_j \right).$$

Also, if $t = \frac{1}{k}$, $a_j = \alpha_j - \alpha_{j+1}$, and $a_k = \alpha_k$, where α_1, α_2,$\cdots$, α_k, are given constant matrices of order $m \times d$, and if $t_j = jt$, for $j = 1, 2, \cdots, k$, then we have easily

$\mu_n = t \left(\sum_{j=1}^{k} \alpha_j \right) \cdot \left(\frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{ni} \sum_{i=1}^{k_n} \eta_{ni}^t \right)$, and

$\sigma_n^2 = \left(\sum_{j=1}^{k} \alpha_j \overrightarrow{\bullet} \left(\frac{1}{k_n} s^2 \left(\underline{\xi}_{k_n} \right) \overrightarrow{\star} s^2 \left(\underline{\eta}_{k_n} \right) \right) \overleftarrow{\bullet} \alpha_j^t \right) t$. These results are involved in the proof of theorem 3.3.9. □

Throughout the proofs of the coming limit theorems in this section we need to represent some Partial sums as path integrals. Let ξ_{n1}, ξ_{n2},$\cdots$,ξ_{nk_n}, and η_{n1}, η_{n2},$\cdots$,η_{nk_n}, be random arrays from $(\Omega, \mathcal{A}, P)$ to $(\mathbb{R}^{d_1}, \mathbb{B}^{d_1})$, and $(\mathbb{R}^{d_2}, \mathbb{B}^{d_2})$ respectively, and $S_n(t) := \sum_{i=1}^{[k_n t]} \xi_{ni}\eta_{ni}^t$, and let ω_{ni}, $i = 1, 2, \cdots, k_n$, $n \in \mathbb{N}$, be constant matrices of order $m \times d_1$. One can write

$$\sum_{i=1}^{k_n} \omega_{ni}\xi_{ni}\eta_{ni}^t = \int_0^1 \omega_n(t) dS'(t),$$

where $S_n'(t)$ is defined by

$$S_n'(t) = S_n\left(\frac{i-1}{k_n}\right) + k_n\left(t - \frac{i-1}{k_n}\right)\left(S_n\left(\frac{i}{k_n}\right) - S_n\left(\frac{i-1}{k_n}\right)\right),$$

where $i := [k_n t] + 1$ in this formula, and $\omega_n \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d_1})$ is defined such that $\omega_{ni} = k_n \int_{\frac{i-1}{k_n}}^{\frac{i}{k_n}} \omega_n(t)dt$, for $i = 1, \cdots, k_n$. Also, we need to see the behavior of $\parallel \sum\limits_{i=1}^{k_n} \omega_{ni}\xi_{ni}\eta_{ni}^t \parallel$ when $\parallel \omega_n \parallel_2$ tends to zero with n tends to infinity, the next lemma gives an answer to that question.

Lemma 3.3.8. If the random pairs (ξ_{n1}, η_{n1}), (ξ_{n2}, η_{n2}), $\cdots$, $(\xi_{nk_n}, \eta_{nk_n})$, are H_2-exchangeable under $P(\cdot|\mathcal{C}_n)$, and the sequences $\left(\frac{1}{\sqrt{k_n}} \sum\limits_{i=1}^{k_n} \xi_{ni}\right)_{n\in\mathbb{N}}$, $\left(\frac{1}{\sqrt{k_n}} \sum\limits_{i=1}^{k_n} \eta_{ni}\right)_{n\in\mathbb{N}}$, $\left(\frac{1}{\sqrt{k_n}} s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$, and $\left(\frac{1}{\sqrt{k_n}} s^2(\underline{\eta}_{k_n})\right)_{n\in\mathbb{N}}$, are stochastically bounded, then

$$\parallel \sum_{i=1}^{k_n} \omega_{ni}\xi_{ni}\eta_{ni}^t \parallel \xrightarrow{P} 0, \text{ if } \parallel \omega_n \parallel_2 \underset{n\longrightarrow\infty}{\longrightarrow} 0.$$

Proof: Since the sequences $\left(\frac{1}{\sqrt{k_n}} \sum\limits_{i=1}^{k_n} \xi_{ni}\right)_{n\in\mathbb{N}}$, and $\left(\frac{1}{\sqrt{k_n}} \sum\limits_{i=1}^{k_n} \eta_{ni}\right)_{n\in\mathbb{N}}$, are stochastically bounded, it is sufficient to prove

$$\parallel \sum_{i=1}^{k_n} \omega_{ni} \left(\xi_{ni} - \bar{\xi}_n\right) \left(\eta_{ni} - \bar{\eta}_n\right)^t \parallel \xrightarrow{P} 0, \text{ if } \parallel \omega_n \parallel_2 \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

where as we know $\bar{\xi}_n := \frac{1}{k_n} \sum\limits_{i=1}^{k_n} \xi_{ni}$, and $\bar{\eta}_n := \frac{1}{k_n} \sum\limits_{i=1}^{k_n} \eta_{ni}$.
Since $\omega \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d_1})$, and from Lemma 3.3.3, we have

$$Var\left(\left(\sum_{i=1}^{k_n} \omega_{ni} \left(\xi_{ni} - \bar{\xi}_n\right) \left(\eta_{ni} - \bar{\eta}_n\right)^t\right) \middle| \mathcal{S}^2\left(\underline{\xi, \eta}_{k_n}, \mathcal{C}_n\right)\right) =$$
$$= \frac{1}{k_n} \sum_{i=1}^{k_n} \omega_{ni} \overrightarrow{\bullet} \left(\frac{1}{k_n} s^2(\underline{\xi}_{k_n}) \overrightarrow{\star} s^2(\underline{\eta}_{k_n})\right) \overleftarrow{\bullet} \omega_{ni}^t + o_P(1).$$

Now, since

$$\frac{1}{k_n} \sum_{i=1}^{k_n} \parallel \omega_{ni} \parallel^2 \leqslant \int_0^1 \parallel \omega_n \parallel^2 dt = \parallel \omega_n \parallel_2^2,$$

and since the sequences $\left(\frac{1}{\sqrt{k_n}} s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$, and $\left(\frac{1}{\sqrt{k_n}} s^2(\underline{\eta}_{k_n})\right)_{n\in\mathbb{N}}$ are stochastically bounded, we obtain

$$Var\left(\left(\sum_{i=1}^{k_n} \omega_{ni} \left(\xi_{ni} - \bar{\xi}_n\right) \left(\eta_{ni} - \bar{\eta}_n\right)^t\right) \middle| \mathcal{S}^2\left(\underline{\xi, \eta}_{k_n}, \mathcal{C}_n\right)\right) \xrightarrow{P} 0.$$

Therefore,

$$E\left(\parallel \sum_{i=1}^{k_n} \omega_{ni} \left(\xi_{ni} - \bar{\xi}_n\right) \left(\eta_{ni} - \bar{\eta}_n\right)^t \parallel^2 \middle| \mathcal{S}^2\left(\underline{\xi, \eta}_{k_n}, \mathcal{C}_n\right)\right) \xrightarrow{P} 0,$$

and from theorem 1.1.3 we conclude that the assertion is valid indeed. □

In the next theorem we discuss the asymptotic normality of the sum $\sum_{i=1}^{k_n} \omega_{ni}\xi_{ni}\eta_{ni}^t$.

Theorem 3.3.9. Suppose that the triangular array $\underline{\xi, \eta}_{k_n}$ is infinitesimal and that the sequences $\left(\frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \xi_{ni}\right)_{n\in\mathbb{N}}$, $\left(\frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \eta_{ni}\right)_{n\in\mathbb{N}}$, $\left(\frac{1}{\sqrt{k_n}} s^2(\underline{\xi}_{k_n})\right)_{n\in\mathbb{N}}$, and $\left(\frac{1}{\sqrt{k_n}} s^2(\underline{\eta}_{k_n})\right)_{n\in\mathbb{N}}$ are stochastically bounded, and further that sequence $(\omega_n)_{n\in\mathbb{N}}$ is a relatively compact sequence in the space $\mathcal{L}^2([0,1], \mathcal{M}_{m\times d_1})$. We call it a sequence of weight functions and assume that these weights are constants. If $\mathcal{C}_n \supseteq \mathcal{S}^2\left(\underline{\xi, \eta}_{k_n}\right)$, $n \in \mathbb{N}$, is a sequence of sub-σ-fields of $\mathcal{A}$, and if the rows of the triangular array are H_2-exchangeable under $P(\cdot|\mathcal{C}_n)$, then we have

$$P * \sum_{i=1}^{k_n} \omega_{ni}\xi_{ni}\eta_{ni}^t \sim \mathcal{N}(\mu_n, \sigma_n^2)\ (\mathcal{C}_n),$$

where

$$\mu_n := \int_0^1 \omega_n(t)dt \cdot \frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{ni} \sum_{i=1}^{k_n} \eta_{ni}^t,$$

and

$$\sigma_n^2 := \int_0^1 \omega_n(t) \overrightarrow{\bullet} \left(\frac{1}{k_n} s^2(\underline{\xi}_{k_n}) \overrightarrow{\star}\, s^2(\underline{\eta}_{k_n})\right) \overleftarrow{\bullet}\, \omega_n^t(t)dt.$$

Proof: Since, $(\omega_n(t))_{n\in\mathbb{N}}$ is relatively compact, so every subsequence of $(\omega_n(t))_{n\in\mathbb{N}}$ contains another convergent sub-subsequence, and consequently it is sufficient to prove the assertion in the case when the sequence $(\omega_n(t))_{n\in\mathbb{N}}$ is convergent. And from lemma 3.3.8 we conclude that it is sufficient to prove it in the case $\omega_n(t) = \omega(t)$, $\forall n \in \mathbb{N}$. Moreover, remarking that any $\omega \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d_1})$ can assumed as a limit of some sequence of step-functions belong to $M_{m\times d_1}$, where $M_{m\times d_1} := \left\{\omega : \omega(t) = \sum_{i=1}^{k} x_i \cdot 1_{[\frac{i-1}{k}, \frac{i}{k})}(t), x_i \in \mathcal{M}_{m\times d_1},\ i = 1, \cdots, k,\ k \in \mathbb{N}\right\}$. By applying lemma 3.3.8 we find that it is sufficient to prove it when $\omega \in M_{m\times d_1}$. Now, we note that

$\sum_{i=1}^{k_n} \omega_{ni}\xi_{ni}\eta_{ni}^t = \int_0^1 \omega(t)dS_n'(t) = \sum_{i=1}^{k} x_i\left(S_n'\left(\frac{i}{k}\right) - S_n'\left(\frac{i-1}{k}\right)\right) = \sum_{i=1}^{k} a_i S_n'\left(\frac{i}{k}\right)$, where $a_i := x_i - x_{i+1},\ i = 1, 2, \cdots, k-1,\ a_k = x_k$. Since the triangular array $\underline{\xi}_{k_n}$ is infintesimal, and from remark 3.3.6, and corollary 3.3.7, we find that the assertion is valid indeed. □

Remark 3.3.10. In theorem 3.3.9, σ_n^2 can be rewritten as $\sigma_n^2 = \frac{1}{k_n}\sum_{i=1}^{k_n} \omega_{ni} \overrightarrow{\bullet} \left(\frac{1}{k_n} s^2(\underline{\xi}_{k_n}) \overrightarrow{\star} s^2(\underline{\eta}_{k_n})\right) \overleftarrow{\bullet}\ \omega_{ni}^t,\ n \in \mathbb{N}$. □

We want at this position to introduce some important properties of the concepts of the H_2-exchangeablity under $P(\cdot|\mathcal{C})$, where $\mathcal{C}$ is a given sub-σ-field of $\mathcal{A}$, also we want to discuss some properties of the σ-field $\mathcal{S}^2\left(\underline{\xi,\eta}_k\right)$, $k \in \mathbb{N}$. For this, we shall introduce the definition of the H_2-exchangeablity under P, also some interesting results.

Definition 3.3.11. Under the same hypotheses of definition 3.3.1, we call the triangular array $\underline{\xi,\eta}_{k_n}$ H_2-exchangeable under P iff this triangular array is H_2-exchangeable under $P(\cdot|\mathcal{C})$, for $\mathcal{C} = \{\emptyset, \Omega\}$.

– It is clear from lemma 3.3.3 that the H_2-exchangeablity under $P(\cdot|\mathcal{C})$, for all $\mathcal{C} \subseteq \mathcal{S}^2\left(\underline{\xi,\eta}_{k_n}\right)$ is equivalent to the H_2-exchangeablity under P.

Lemma 3.3.12. Let $(X_1, Y_1),\ (X_2, Y_2),\ \cdots,\ (X_k, Y_k)$ be random pairs from $(\Omega, \mathcal{A}, P)$ to $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. Let further, $f, g : \mathcal{E} \times \mathcal{E}^k \longrightarrow \mathbb{R}^{d_1}, \mathbb{R}^{d_2}$ respectively be measurable functions, each of them is permutation symmetric in its second variable, and let $\xi_i := f(X_i, \underline{X}_k),\ \eta_i := g(Y_i, \underline{Y}_k),\ i = 1, \cdots, k$, then $\mathcal{S}^2\left(\underline{\xi,\eta}_k\right) \subseteq \mathcal{S}^2\left(\underline{X,Y}_k\right)$.

Proof: It is sufficient to prove that the following implication

$$l\left(\underline{\xi,\eta}_k\right) \text{ is } \mathcal{S}^2\left(\underline{\xi,\eta}_k\right)\text{-measurable} \Longrightarrow l\left(\underline{\xi,\eta}_k\right) \text{ is } \mathcal{S}^2\left(\underline{X,Y}_k\right)\text{-measurable}$$

is valid for all measurable $l : \left(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\right)^k \longrightarrow \mathbb{R}$.
For this, let $h : (\mathcal{E}^2)^k \longrightarrow \left(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\right)^k$ be defined by

$$h(\underline{x,y}_k) := \left(\left(f(x_1, \underline{x}_k), g(y_1, \underline{y}_k)\right), \cdots, \left(f(x_k, \underline{x}_k), g(y_k, \underline{y}_k)\right)\right)$$

Let $l\left(\underline{\xi,\eta}_k\right)$ be $\mathcal{S}^2\left(\underline{\xi,\eta}_k\right)$-measurable, this implies $l\left(\lambda\left(\underline{\xi,\eta}_k\right)\right)=l\left(\underline{\xi,\eta}_k\right)$ for all $\lambda\in\Lambda_k$. Let us define $l_0:(\mathcal{E}^2)^k\longrightarrow\mathbb{R}$ by $l_0\left(\underline{x,y}_k\right):=l(h(\underline{x,y}_k))$. Hence, $l_0\left(\underline{X,Y}_k\right)=l\left(h\left(\underline{X,Y}_k\right)\right)=l\left(\underline{\xi,\eta}_k\right)$, and $l_0\left(\lambda\left(\underline{X,Y}_k\right)\right)=$
$=l\left(h\left(\lambda\left(\underline{X,Y}_k\right)\right)\right)=l\left(\lambda\left(\underline{\xi,\eta}_k\right)\right)=l\left(\underline{\xi,\eta}_k\right)=l_0\left(\underline{X,Y}_k\right)$.
Hence, $l_0\left(\lambda\left(\underline{X,Y}_k\right)\right)=l_0\left(\underline{X,Y}_k\right)$, but this means that $l_0\left(\underline{X,Y}_k\right)$ is $\mathcal{S}^2\left(\underline{X,Y}_k\right)$-measurable, and also $l\left(\underline{\xi,\eta}_k\right)$ is $\mathcal{S}^2\left(\underline{X,Y}_k\right)$-measurable. Consequently, the assertion is valid indeed. □

Lemma 3.3.13. Let $f,g:\mathcal{E}\times\mathcal{E}^k\longrightarrow\mathbb{R}^{d_1},\mathbb{R}^{d_2}$ be measurable functions, each of them is permutation symmetric in its second variable. If the random pairs $(X_1,Y_1),\cdots,(X_k,Y_k)$ are H_2-exchangeable under $P(\cdot|\mathcal{C})$, then $(f(X_1,\underline{X}_k),g(Y_1,\underline{Y}_k)),\cdots,(f(X_k,\underline{X}_k),g(Y_k,\underline{Y}_k))$, are also H_2-exchangeable under $P(\cdot|\mathcal{C})$.

Proof: It is sufficient to prove that for any measurable function $l:\left(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}\right)^k\longrightarrow\mathbb{R}$ satisfying that the quantity
$E\left(l\left(\left(f\left(X_1,\underline{X}_k\right),g\left(Y_1,\underline{Y}_k\right)\right),\cdots,\left(f\left(X_k,\underline{X}_k\right),g\left(Y_k,\underline{Y}_k\right)\right)\right)\right)$
is well-defined, the equality

$$E\left(l\left(\lambda\left(\left(f\left(X_1,\underline{X}_k\right),g\left(Y_1,\underline{Y}_k\right)\right),\cdots,\left(f\left(X_k,\underline{X}_k\right),g\left(Y_k,\underline{Y}_k\right)\right)\right)\right)|\mathcal{C}\right)=$$
$$=E\left(l\left(\left(f\left(X_1,\underline{X}_k\right),g\left(Y_1,\underline{Y}_k\right)\right),\cdots,\left(f\left(X_k,\underline{X}_k\right),g\left(Y_k,\underline{Y}_k\right)\right)\right)|\mathcal{C}\right)\ [P]$$

is valid. For this, let us define $h:(\mathcal{E}^2)^k\longrightarrow\left(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}\right)^k$ by
$h(\underline{x,y}_k)=\left(\left(f\left(x_1,\underline{x}_k\right),g\left(y_1,\underline{y}_k\right)\right),\cdots,\left(f\left(x_k,\underline{x}_k\right),g\left(y_k,\underline{y}_k\right)\right)\right)^t$. And we can write
$h(\underline{X,Y}_k)=\left(\left(f\left(X_1,\underline{X}_k\right),g\left(Y_1,\underline{Y}_k\right)\right),\cdots,\left(f\left(X_k,\underline{X}_k\right),g\left(Y_k,\underline{Y}_k\right)\right)\right)^t$. Consequently, it remains to be proved the validity of the equality

$$E\left(l\left(\lambda\left(h\left(\underline{X,Y}_k\right)\right)\right)|\mathcal{C}\right)=E\left(l\left(h\left(\underline{X,Y}_k\right)\right)|\mathcal{C}\right)\ [P].$$

But easily, we see that the left side is equal to $E\left(l\left(h\left(\lambda\left(\underline{X,Y}_k\right)\right)\right)|\mathcal{C}\right)$ almost sure with respect to P. Since $(X_1,Y_1),\cdots,(X_k,Y_k)$ are H_2-exchangeable under $P(\cdot|\mathcal{C})$, the assertion follows. □

Lemma 3.3.14. Let $(\xi_1, \eta_1), \cdots, (\xi_k, \eta_k) : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ be random pairs, and $\mathcal{C} \subseteq \mathcal{A}$ be a sub-σ-field.
Let $\underline{x, y}_k = ((x_1, y_1), \cdots, (x_k, y_k)) \in (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})^k$, $\bar{x} = \frac{1}{k} \sum_{i=1}^{k} x_i$, $\bar{y} = \frac{1}{k} \sum_{i=1}^{k} y_i$, and $V(\underline{x}_k) = \sum_{i=1}^{k} (x_i - \bar{x})(x_i - \bar{x})^t$ $V\left(\underline{y}_k\right) = \sum_{i=1}^{k} (y_i - \bar{y})(y_i - \bar{y})^t$. Assume that $V(\underline{x}_k)$, and $V\left(\underline{y}_k\right)$ are positive definite, and let $h_1 : \mathbb{R}^{d_1} \times (\mathbb{R}^{d_1})^k \longrightarrow \mathbb{R}^{d_1}$ be defined by $h_1(y, \underline{x}_k) = (V(\underline{x}_k))^{-\frac{1}{2}} (y - \bar{x})$, and let $h_2 : \mathbb{R}^{d_2} \times (\mathbb{R}^{d_2})^k \longrightarrow \mathbb{R}^{d_2}$ be defined by $h_2(y, \underline{x}_k) = (V(\underline{x}_k))^{-\frac{1}{2}} (y - \bar{x})$, and $\tilde{\xi}_i = h_1\left(\xi_i, \underline{\xi}_k\right)$, $\tilde{\eta}_i = h_2\left(\eta_i, \underline{\eta}_k\right)$, $i = 1, \cdots, k$. If (ξ_1, η_1), $\cdots$, (ξ_k, η_k) are H_2-exchangeable under $P(\cdot|\mathcal{C})$, and if $\bar{\xi}$, $\bar{\eta}$, $V(\underline{\xi}_k)$, and $V(\underline{\eta}_k)$ are $\mathcal{C}$-measurable, then $\mathcal{S}^2\left(\underline{\xi, \eta}_k, \mathcal{C}\right) = \mathcal{S}^2\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \mathcal{C}\right)$.

Proof: First we see that $\mathcal{S}^2\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \mathcal{C}\right) \subseteq \mathcal{S}^2\left(\underline{\xi, \eta}_k, \mathcal{C}\right)$ follows from lemma 3.3.12 because h_1, h_2 are permutation symmetric in their second variable. It remains to show that $\mathcal{S}^2\left(\underline{\xi, \eta}_k, \mathcal{C}\right) \subseteq \mathcal{S}^2\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \mathcal{C}\right)$. And for this it is sufficient to prove that $\mathcal{S}^2\left(\underline{\xi, \eta}_k\right) \subseteq \mathcal{S}^2\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \mathcal{C}\right)$. Let $f : (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})^k \longrightarrow \mathbb{R}$ be any measurable function such that $f\left(\underline{\xi, \eta}_k\right)$ is $\mathcal{S}^2\left(\underline{\xi, \eta}_k\right)$-measurable. This means $f\left(\lambda\left(\underline{\xi, \eta}_k\right)\right) = f\left(\underline{\xi, \eta}_k\right)$ for all $\lambda \in \Lambda_k$.
To complete this proof we have to prove that $f\left(\underline{\xi, \eta}_k\right)$ is $\mathcal{S}^2\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \mathcal{C}\right)$-measurable. For this, let $z = ((z_1, \tilde{z}_1), \cdots, (z_k, \tilde{z}_k)) \in (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})^k$, $a \in \mathbb{R}^{d_1}$, $b \in \mathbb{R}^{d_2}$, $A \in \mathcal{M}_{d_1 \times d_1}$, $B \in \mathcal{M}_{d_2 \times d_2}$, and the matrices A, B are positive definite and symmetric, h be defined by $h(z, a, b, A, B) = ((A^{\frac{1}{2}} z_1 + a, B^{\frac{1}{2}} \tilde{z}_1 + b), \cdots, (A^{\frac{1}{2}} z_k + a, B^{\frac{1}{2}} \tilde{z}_k + b))^t$. It is easy to see that $h(\lambda(z), a, b, A, B) = \lambda(h(z, a, b, A, B))$, $\underline{\xi, \eta}_k = h(\underline{\tilde{\xi}, \tilde{\eta}}_k, \bar{\xi}, \bar{\eta}, V(\underline{\xi}_k), V(\underline{\eta}_k))$, and
$f\left(\underline{\xi, \eta}_k\right) = f\left(h\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \bar{\xi}, \bar{\eta}, V(\underline{\xi}_k), V(\underline{\eta}_k)\right)\right)$ but $\bar{\xi}$, $\bar{\eta}$, $V\left(\underline{\xi}_k\right)$ and $V\left(\underline{\eta}_k\right)$ are $\mathcal{S}^2\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \mathcal{C}\right)$-measurable because $\mathcal{C} \subseteq \mathcal{S}^2\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \mathcal{C}\right)$, and they are $\mathcal{C}$-measurable. But also,

$$f\left(h\left(\lambda\left(\underline{\tilde{\xi}, \tilde{\eta}}_k\right), \bar{\xi}, \bar{\eta}, V(\underline{\xi}_k), V(\underline{\eta}_k)\right)\right) = f\left(h\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \bar{\xi}, \bar{\eta}, V(\underline{\xi}_k), V(\underline{\eta}_k)\right)\right),$$

for all $\lambda \in \Lambda_k$.
Thus, $f\left(\underline{\xi, \eta}_k\right) = f\left(h\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \bar{\xi}, \bar{\eta}, V(\underline{\xi}_k), V(\underline{\eta}_k)\right)\right)$ is $\mathcal{S}^2\left(\underline{\tilde{\xi}, \tilde{\eta}}_k, \mathcal{C}\right)$-measurable. □

In the introduction of this section we have defined the statistic $T_n = \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \omega_{ni} f_n\left(X_{ni}, \underline{X}_{k_n}\right)\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right)\right)^t$, we want here to make some preliminary computations to prepare for the next theorem which will discuss the conditional asymptotic normality of the sequence $(T_n)_{n\in\mathbb{N}}$. For this, let us first prove that $E\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right)\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right)\right)^t |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)$, and $Var\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right)\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right)\right)^t |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)$, are independent of i.
One can easily see

$$E\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right)\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right)\right)^t |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right) = \\ = \frac{1}{k_n^2} \sum_{i=1}^{k_n} f_n\left(X_{ni}, \underline{X}_{k_n}\right) \sum_{i=1}^{k_n} \left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right)\right)^t \ [\mathbb{P}],$$

$$Var\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right) |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right) = \\ = \frac{1}{k_n} \sum_{i=1}^{k_n} \left(f_n\left(X_{ni}, \underline{X}_{k_n}\right) - E\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right) |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right) \cdot \\ \cdot \left(f_n\left(X_{ni}, \underline{X}_{k_n}\right) - E\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right) |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right)^t\right) \ [\mathbb{P}],$$

$$Var\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right) |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right) = \\ = \frac{1}{k_n} \sum_{i=1}^{k_n} \left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right) - E\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right) |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right) \cdot \\ \cdot \left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right) - E\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right) |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right)^t\right) \ [\mathbb{P}],$$

Therefore, for notational convenience we denote
$E\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right) |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)$ by $E\left(f_n |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)$, i.e.

$$E\left(f_n |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right) := E\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right) |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right),$$

Also, we denote similarly
$E\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right) |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)$ by $E\left(g_n |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)$, i.e.

$$E\left(g_n |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right) := E\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right) |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right),$$

Also, we define $Var\left(f_n |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)$, and $Var\left(g_n |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)$. We can see similarly that $Var\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right)\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right)\right)^t |\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)$ is independent of i.
But we want here to have an explicit expression for it. For this, we begin

with
$$Var\left(\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)\cdot \right.$$
$$\left. \cdot\left(g_n\left(Y_{ni},\underline{Y}_{k_n}\right)-E\left(g_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)^t|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right).$$
We mention here that the computations are made only in the case where $\left(Var\left(f_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)_{n\in\mathbb{N}}$, and $\left(Var\left(g_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)_{n\in\mathbb{N}}$ are stochastically bounded.
$$Var\left(\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)-E\left(f_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)\cdot \right.$$
$$\left. \cdot\left(g_n\left(Y_{ni},\underline{Y}_{k_n}\right)-E\left(g_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)^t|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)=$$
$$=Var\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\overrightarrow{\star}Var\left(g_n\left(Y_{ni},\underline{Y}_{k_n}\right)|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\ [\mathbb{P}]$$
$$=\left(\frac{1}{k_n}\sum_{i=1}^{k_n}\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)-E\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)\cdot \right.$$
$$\left.\cdot\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)-E\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)^t\right)\overrightarrow{\star}$$
$$\overrightarrow{\star}\left(\frac{1}{k_n}\sum_{i=1}^{k_n}\left(g_n\left(Y_{ni},\underline{Y}_{k_n}\right)-E\left(g_n\left(Y_{ni},\underline{Y}_{k_n}\right)|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)\cdot \right.$$
$$\left.\cdot\left(g_n\left(Y_{ni},\underline{Y}_{k_n}\right)-E\left(g_n\left(Y_{ni},\underline{Y}_{k_n}\right)|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)^t\right)\ [\mathbb{P}].$$
Now, if $\left(\sqrt[4]{k_n}E\left(f_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)_{n\in\mathbb{N}}$, and $\left(\sqrt[4]{k_n}E\left(g_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)_{n\in\mathbb{N}}$ are also stochastically bounded, then we have
$$Var\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)\left(g_n\left(Y_{ni},\underline{Y}_{k_n}\right)\right)^t|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)=$$
$$=Var\left(f_n\left(X_{ni},\underline{X}_{k_n}\right)|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\overrightarrow{\star}Var\left(g_n\left(Y_{ni},\underline{Y}_{k_n}\right)|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)+o_p(1).$$
Let us now give explicit expressions for $E\left(T_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)$, and $Var\left(T_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)$.
$$E\left(T_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)=$$
$$=\frac{1}{\sqrt{k_n}}\sum_{i=1}^{k_n}\omega_{ni}E\left(f_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\left(E\left(g_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)^t\ [\mathbb{P}],$$
$$Var\left(T_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)=$$
$$=Var\left(\frac{1}{\sqrt{k_n}}\sum_{i=1}^{k_n}\omega_{ni}f_n\left(X_{ni},\underline{X}_{k_n}\right)\left(g_n\left(Y_{ni},\underline{Y}_{k_n}\right)\right)^t|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\ [\mathbb{P}]$$
$$=\frac{1}{k_n}\sum_{i=1}^{k_n}\omega_{ni}\overrightarrow{\bullet}\left(Var\left(f_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\overrightarrow{\star}Var\left(g_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)\overleftarrow{\bullet}\omega_{ni}^t+o_p(1).$$

– For the next theorem we assume first the same situation in the introduction of this section.

Theorem 3.3.15. Assume that the triangular arrays

$$\frac{1}{\sqrt[4]{k_n}} f_n\left(X_{ni}, \underline{X}_{k_n}\right), \ \frac{1}{\sqrt[4]{k_n}} g_n\left(Y_{ni}, \underline{Y}_{k_n}\right), \ 1 \leqslant i \leqslant k_n, \ n \in \mathbb{N},$$

are infinitesimal and the following sequences

$$\left(Var\left(f_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)_{n\in\mathbb{N}}, \left(Var\left(g_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)_{n\in\mathbb{N}},$$
$$\left(\sqrt[4]{k_n} E\left(f_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)_{n\in\mathbb{N}}, \text{ and } \left(\sqrt[4]{k_n} E\left(g_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)_{n\in\mathbb{N}}$$

are stochastically bounded. Assume further that the sequence $(\omega_n(t))_{n\in\mathbb{N}} \subseteq \mathcal{L}^2([0,1], \mathcal{M}_{m\times d_1})$ is relatively compact sequence of non-random weight functions. Then the sequence $(T_n)_{n\in\mathbb{N}}$ is asymptotically normal under $\mathbb{P} \in H_2$, and conditioned by the symmetric sub-σ-field $\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)$. More explicitly,

$$\mathbb{P} * \left(T_n - E\left(T_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right) \sim$$
$$\sim \mathcal{N}\left(0, Var\left(T_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right) \ \left(\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right),$$

also

$$\mathbb{P} * T_n \sim \mathcal{N}\left(E\left(T_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right), Var\left(T_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right) \ \left(\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right).$$

Proof: Let ξ_{ni}, η_{ni}, $i = 1, 2, \cdots, k_n$, $n \in \mathbb{N}$, be defined by

$$\xi_{ni} := \frac{1}{\sqrt[4]{k_n}}\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right) - E\left(f_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right), \ 1 \leqslant i \leqslant k_n, \ n \in \mathbb{N},$$

and

$$\eta_{ni} := \frac{1}{\sqrt[4]{k_n}}\left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right) - E\left(g_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right), \ 1 \leqslant i \leqslant k_n, \ n \in \mathbb{N}.$$

From lemma 3.3.13, the random pairs $(\ \xi_{n1}, \ \eta_{n1}\ ), \cdots, (\ \xi_{nk_n}, \ \eta_{nk_n}\ )$, are H_2-exchangeable under $\mathbb{P}\left(\cdot, \mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)$. Let

$$F_n(t) := \sum_{i=1}^{[k_n t]} \frac{1}{\sqrt{k_n}}\left(f_n\left(X_{ni}, \underline{X}_{k_n}\right) - E\left(f_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right) \cdot$$
$$\cdot \left(g_n\left(Y_{ni}, \underline{Y}_{k_n}\right) - E\left(g_n|\mathcal{S}^2\left(\underline{X,Y}_{k_n}\right)\right)\right)^t,$$

$0 \leqslant t \leqslant 1$, and let

$$F_n'(t) := F_n\left(\frac{i-1}{k_n}\right) + k_n\left(t - \frac{i-1}{k_n}\right)\left(F_n\left(\frac{i}{k_n}\right) - F_n\left(\frac{i-1}{k_n}\right)\right),$$

where $i := [k_n t] + 1$.
Therefore, we can write

$$\sum_{i=1}^{k_n} \omega_{ni} \xi_{ni} \eta_{ni}^t = \int_0^1 \omega_n(t) dF_n'(t).$$

Hence, all assumption of theorem 3.3.9 are satisfied, and consequently we have

$$\mathbb{P} * \sum_{i=1}^{k_n} \omega_{ni} \xi_{ni} \eta_{ni}^t \sim \mathcal{N}\left(0, \sigma_n^2\right) \ \left(\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right).$$

where $\sigma_n^2 := \int_0^1 \omega_n(t) \overrightarrow{\bullet} \left(\frac{1}{k_n} s^2\left(\underline{\xi}_{k_n}\right) \overrightarrow{\star} s^2\left(\underline{\eta}_{k_n}\right)\right) \overleftarrow{\bullet} \omega_n^t(t) dt =$
$= \int_0^1 \omega_n(t) \overrightarrow{\bullet} \left(Var\left(f_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right) \overrightarrow{\star} Var\left(g_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right) \overleftarrow{\bullet} \omega_n^t(t) dt.$
Also, from remark 3.3.10 we can write
$\sigma_n^2 = \frac{1}{k_n} \sum_{i=1}^{k_n} \omega_{ni} \overrightarrow{\bullet} \left(\frac{1}{k_n} s^2(\underline{\xi}_{k_n}) \overrightarrow{\star} s^2(\underline{\eta}_{k_n})\right) \overleftarrow{\bullet} \omega_{ni}^t$
$\sigma_n^2 = \frac{1}{k_n} \sum_{i=1}^{k_n} \omega_{ni} \overrightarrow{\bullet} \left(Var\left(f_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right) \overrightarrow{\star} Var\left(g_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right) \overleftarrow{\bullet} \omega_{ni}^t \ [\mathbb{P}].$
By remarking that $\sigma_n^2 - Var\left(T_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right) \xrightarrow{\mathbb{P}} 0$, we conclude that

$$\mathbb{P} * \sum_{i=1}^{k_n} \omega_{ni} \xi_{ni} \eta_{ni}^t \sim \mathcal{N}\left(0, Var\left(T_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right) \ \left(\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right).$$

is valid.
Now, if the sequences
$\left(\sqrt[4]{k_n} E\left(f_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right)_{n \in \mathbb{N}}$, and $\left(\sqrt[4]{k_n} E\left(g_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right)_{n \in \mathbb{N}}$
are stochastically bounded, then we have
$\mathbb{P} * \left(T_n - E\left(T_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right) \sim$
$\sim \mathcal{N}\left(0, Var\left(T_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right) \ \left(\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right),$
and

$$\mathbb{P} * T_n \sim \mathcal{N}\left(E\left(T_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right), Var\left(T_n | \mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right)\right) \ \left(\mathcal{S}^2\left(\underline{X, Y}_{k_n}\right)\right).$$

□

3.4. The Hypothesis H_3 (random blocks):

Introduction: Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathbb{P}$ be a probability measure defined on $\mathcal{A}$, $\underline{X}_n = (X_{n11}, X_{n12}, \cdots, X_{n1k}; X_{n21}, X_{n22}, \cdots, X_{n2k}; \cdots; X_{nn1}, X_{nn2}, \cdots, X_{nnk})^t$, $n \in \mathbb{N}$, be a triangular array of random elements with values in a sample space $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. Assume that the random elements of each row $\Big($ i.e. $X_{n11}, X_{n12}, \cdots, X_{n1k}; X_{n21}, X_{n22}, \cdots, X_{n2k}; \cdots; X_{nn1}, X_{nn2}, \cdots, X_{nnk}, n \in \mathbb{N} \Big)$ are independent under $\mathbb{P}$, and the random elements of each block are identically distributed under $\mathbb{P}$ also. Let the distribution of X_{nij} under $\mathbb{P}$ be denoted by $\mathbb{P} * X_{nij}$ or shortly by P_{ni}. The general form of linear statistics which are considered here is denoted by $T_{nj}(\underline{X}_n)$, or shortly by T_{nj}, $j = 1, \cdots, k$, if we can avoid confusion, where $T_{nj} := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \omega_{ni} f_n(X_{nij}, \underline{X}_n)$, and where $\omega_{ni} \in \mathcal{M}_{m \times d}$, $i = 1, \cdots, n$, $n \in \mathbb{N}$, where the matrices ω_{ni}, $i = 1, \cdots, n$, $n \in \mathbb{N}$, are assumed to be given constants. We define here also a new concept of symmetry
w depends on $\underline{x}_n$ in a permutation symmetric way iff

$$w(\theta(\underline{x}_n)) = w(\underline{x}_n), \forall\, \theta \in \Theta_n.$$

We mention here that this is the symmetry concept, which is used in all arguments related with this section, and the symbols θ, Θ_n are introduced just before definition 3.4.1. The functions $f_n : \mathcal{E} \times \mathcal{E}^{kn} \longrightarrow \mathbb{R}^d$, $(x, \underline{y}_n) \longmapsto f_n(x, \underline{y}_n)$, $n \in \mathbb{N}$, are measurable and such that they depend on $\underline{y}_n$ in a permutation symmetric way. Moreover, we assume that $\sum_{j=1}^{k} f_n(X_{nij}, \underline{X}_n)$, and $\sum_{j=1}^{k} f_n(X_{nij}, \underline{X}_n)(f_n(X_{nij}, \underline{X}_n))^t$ are independent of i. Fix $n \in \mathbb{N}$, and $i \in \{1, \cdots, n\}$. Let $\lambda \in \mathbb{R}^d$ be a given constant, and define $\Omega_\lambda \subseteq \Omega$ by

$$\Omega_\lambda := \Big\{ \lambda^t f_n(X_{ni1}, \underline{X}_n) = \lambda^t f_n(X_{ni2}, \underline{X}_n) = \cdots = \lambda^t f_n(X_{nik}, \underline{X}_n) \Big\}$$

we assume that $\mathbb{P}(\Omega_\lambda) > 0$ is hold iff $\lambda = 0$. Now, for notational convenience concerning the computations, let $\omega_n \in \mathcal{L}^2([0,1], \mathcal{M}_{m \times d})$ be such that

$\omega_{ni} = k_n \int\limits_{\frac{i-1}{k_n}}^{\frac{i}{k_n}} \omega_n(t)dt,\ i = 1, \cdots, n,\ n \in \mathbb{N}.$
We mention here that the form of the considered statistic T_{nj} contains the linear rank test statistics $S_j = \sum\limits_{i=1}^{n} a(Rij),\ 1 \leqslant j \leqslant k$, which has been discussed deeply in "Theory of rank tests [1967]" which is due to J. Hájek and Z. Šidák.

To achieve the purpose here we need to build a base for our coming limit theorems, for this we shall begin with the following interesting steps which are needed very much to obtain the final results.

Let us here introduce another new kind of exchangeablity, we call it H_3-exchangeablity, also we put some related symbols, which will be needed for the main results of this section.
Let $(\Omega, \mathcal{A}, P)$ be a probability space and let

$$\underline{\xi}_n = (\xi_{n11}, \cdots, \xi_{n1k}; \xi_{n21}, \cdots, \xi_{n2k}; \cdots; \xi_{nn1}, \cdots, \xi_{nnk})^t$$

be an array of blocks of random elements from $(\Omega, \mathcal{A}, P)$ to $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. And let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence of sub-σ-fields of $\mathcal{A}$, $\mathcal{C}$ be a sub-σ-field of $\mathcal{A}$. Let further $\mathcal{S}^3\left(\underline{\xi}_n\right) := \left(\underline{\xi}_n\right)^{-1}(\mathcal{S}_n^3)$, where $\mathcal{S}_n^3$ denotes here the σ-field of all sets B in $\mathcal{B}^{kn}$ satisfying the condition

$$\underline{x}_n \in B \Longleftrightarrow \theta\left(\underline{x}_n\right) \in B,\ \forall\theta \in \Theta_n,$$

where here $\underline{x}_n := (x_{11}, \cdots, x_{1k}; x_{21}, \cdots, x_{2k}; \cdots; x_{n1}, \cdots, x_{nk})$, and
$\Theta_n := \Big\{\theta : \theta\left(\underline{x}_n\right) = \big(x_{1\pi_1(1)}, \cdots, x_{1\pi_1(k)};\ x_{2\pi_2(1)},\ \cdots,\ x_{2\pi_2(k)};\ \cdots;\ x_{n\pi_n(1)},\ \cdots,\ x_{n\pi_n(k)}\big)$, for some $\pi_1,\ \pi_2,\ \cdots,\ \pi_n$ of the bijective functions defined on $\{1,\ 2,\ \cdots,\ k\}$, $\forall\ \underline{x}_n \in \mathcal{E}^{kn}\Big\}$. Let $s^2(\underline{\xi}_n) := \sum\limits_{j=1}^{k}\left(\xi_{n\cdot j} - \bar{\xi}_n\right)\left(\xi_{n\cdot j} - \bar{\xi}_n\right)^t$, and $\bar{\xi}_n := \frac{1}{k}\sum\limits_{j=1}^{k}\xi_{n\cdot j}$. For each $n \in \mathbb{N}$ the matrix $s^2(\underline{\xi}_{k_n})$ is assumed to be positive definite. Also, all these assumptions are hold in this section.

Definition 3.4.1. The random arrays $\xi_{n11}, \xi_{n12}, \cdots, \xi_{n1k}; \cdots; \xi_{nn1}, \xi_{nn2}, \cdots, \xi_{nnk}$ are H_3-exchangeable under $P(\cdot|\mathcal{C}_n)$, iff $\forall\theta \in \Theta_n$ the equality

$$P\left(\underline{\xi}_n \in B|\mathcal{C}_n\right) = P\left(\theta\left(\underline{\xi}_n\right) \in B|\mathcal{C}_n\right) \ [P]$$

is valid for all $B \in \mathcal{B}^{kn}$, and then the array $\underline{\xi}_n$ is called H_3-exchangeable under $P(\cdot|\mathcal{C}_n)$.

Lemma 3.4.2. Let $f : (\mathcal{E}, \mathcal{B})^{kn} \longrightarrow (\mathbb{R}, \mathbb{B})$ be a measurable function such that $E(f(\underline{\xi}_n))$ is well-defined. If $\underline{\xi}_n$ is H_3-exchangeable under $P(\cdot|\mathcal{C})$, then $\forall A \in \mathcal{S}^3(\underline{\xi}_n),\ C \in \mathcal{C},\ \theta \in \Theta_n$ the following equality is valid

$$E\left(1_{A\cap C} f\left(\theta\left(\underline{\xi}_n\right)\right)\right) = E\left(1_{A\cap C} f\left(\underline{\xi}_n\right)\right).$$

□

Let us denote the σ-field $\sigma(\mathcal{S}^3\left(\underline{\xi}_n\right), \mathcal{C})$ by $\mathcal{S}^3(\underline{\xi}_n, \mathcal{C})$. Now, since $\{A \cap C : A \in \mathcal{S}^3\left(\underline{\xi}_n\right),\ C \in \mathcal{C}\}$ generates the σ-field $\mathcal{S}^3(\underline{\xi}_n, \mathcal{C})$, we have by lemma 3.4.2 the validity of

$$E\left(1_D f\left(\theta\left(\underline{\xi}_n\right)\right)\right) = E\left(1_D f\left(\underline{\xi}_n\right)\right),$$

$\forall D \in \mathcal{S}^3(\underline{\xi}_n, \mathcal{C}),\ \theta \in \Theta_n$.
Therefore, the following lemma is just a consequence of lemma 3.4.2.

Lemma 3.4.3. Let $f : (\mathcal{E}, \mathcal{B})^{kn} \longrightarrow (\mathbb{R}, \mathbb{B})$ be a measurable function such that $E(f(\underline{\xi}_n))$ is well-defined. If $\underline{\xi}_n$ is H_3-exchangeable under $P(\cdot|\mathcal{C})$, then

$$E(f(\underline{\xi}_n)|\mathcal{S}^3\left(\underline{\xi}_n, \mathcal{C}\right)) = \frac{1}{(k!)^n} \sum_{\theta\in\Theta_n} f(\theta(\underline{\xi}_n)) \ [P].$$

Let us here put the following theorem which is an invariance principle, needed to obtain the limit theorems of this section.

Theorem 3.4.4. Suppose that the triangular array $\underline{\xi}_n$ satisfying the following conditions:

For each $n \in \mathbb{N}$, $s^2(\underline{\xi}_n) := \sum_{j=1}^{k}(\xi_{nij}-\bar{\xi}_n)(\xi_{nij}-\bar{\xi}_n)^t$, and $\sum_{j=1}^{k}\xi_{nij}$ are independent of the value of i, where $\bar{\xi}_n := \frac{1}{k}\sum_{j=1}^{k}\xi_{n\cdot j}$, i.e. we can write:

$$\sum_{j=1}^{k}\xi_{nij} = \sum_{j=1}^{k}\xi_{n\cdot j},$$

$$\sum_{j=1}^{k}(\xi_{nij}-\bar{\xi}_n)(\xi_{nij}-\bar{\xi}_n)^t = \sum_{j=1}^{k}(\xi_{n\cdot j}-\bar{\xi}_n)(\xi_{n\cdot j}-\bar{\xi}_n)^t.$$

Moreover, the sequences $\left(\frac{n}{k}s^2(\underline{\xi}_n)\right)_{n\in\mathbb{N}}$, and $\left(\frac{n}{k}\sum_{j=1}^{k}\xi_{n\cdot j}\right)_{n\in\mathbb{N}}$ are stochastically bounded. If $\mathcal{C}_n \supseteq \mathcal{S}^3\left(\underline{\xi}_n\right)$, $n \in \mathbb{N}$, is a sequence of sub-σ-fields of $\mathcal{A}$, $\tilde{\mathcal{C}}_n := \sigma(C \times (C[0,1])^d : C \in \mathcal{C}_n)$, and if the triangular array $\underline{\xi}_n$ is H_3-exchangeable under $P(\cdot|\mathcal{C}_n)$ then for $j = 1, 2, \cdots, k$ we have

$$\tilde{\mathcal{S}}_{nj} \overset{w}{\sim} \tilde{W}\ (\tilde{\mathcal{C}}_n).$$

Where

$$\tilde{\mathcal{S}}_{nj} : \Omega \times (C[0,1])^d \longrightarrow (D[0,1])^d,$$
$$\tilde{\mathcal{S}}_{nj}(\omega_1, \omega_2) := \mathcal{S}_{nj}(\omega_1),$$
$$\left(\mathcal{S}_{nj}(\omega_1)\right)(t) := \sum_{i=1}^{[nt]}\xi_{nij}(\omega_1).$$

And

$$\tilde{W} : \Omega \times (C[0,1])^d \longrightarrow (C[0,1])^d \subset (D[0,1])^d,$$
$$\tilde{W}_t(\omega_1, \omega_2) := \sqrt{\frac{n}{k}}\left(\sqrt{s^2(\underline{\xi}_{k_n})}\right)(\omega_1)W_t(\omega_2) + t\frac{n}{k}\sum_{j=1}^{k}\xi_{n\cdot j}(\omega_1),$$

$\forall t \in [0,1]$, $\forall(\omega_1, \omega_2) \in \Omega \times (C[0,1])^d$, where $W := (W_t)_{0\leqslant t\leqslant 1}$ is a d-dimensional Brownian motion.

Proof: We prove it first in the case where $\sup_n \| \frac{n}{k}s^2(\underline{\xi}_n) \| < +\infty\ [P]$, and $\sup_n \| \frac{n}{k}\sum_{j=1}^{k}\xi_{n\cdot j} \| < +\infty\ [P]$. By lemma 1.2.2 the assertion is equivalent to the validity of the limit

$$\int\limits_{C_n\times(C[0,1])^d} \varphi(\tilde{\mathcal{S}}_n)dP\otimes\mu - \int\limits_{C_n\times(C[0,1])^d} \varphi(\tilde{W})dP\otimes\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

for all sequences $(C_n)_{n\in\mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where $C_n \in \mathcal{C}_n$, and where μ is the Wiener measure defined on $(C[0,1])^d$.
Therefore, let $(C_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of the measurable sets $C_n \in \mathcal{C}_n$, $n\in\mathbb{N}$, we want to prove the validity of the previous limit above, which is equivalent to

$$\int\limits_{C_n} \varphi(\mathcal{S}_n)dP - \int\limits_{C_n}\int\limits_{(C[0,1])^d} \varphi(\tilde{W})dPd\mu \underset{n\longrightarrow\infty}{\longrightarrow} 0$$

$$\Longleftrightarrow \int\limits_{C_n} E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right)dP - \int\limits_{C_n} E\left(\int\limits_{(C[0,1])^d} \varphi(\tilde{W})d\mu\middle|\mathcal{C}_n\right)dP \underset{n\longrightarrow\infty}{\longrightarrow} 0$$

Since the sequence $(C_n)_{n\in\mathbb{N}}$ is arbitrary, the previous assertion is equivalent to

$$E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right) - E\left(\int\limits_{(C[0,1])^d} \varphi(\tilde{W})d\mu\middle|\mathcal{C}_n\right) \overset{P}{\longrightarrow} 0$$

$$\Longleftrightarrow E\left(\varphi(\mathcal{S}_n)|\mathcal{C}_n\right) - \int\limits_{(C[0,1])^d} \varphi(\tilde{W})d\mu \overset{P}{\longrightarrow} 0$$

$$\Longleftrightarrow \frac{1}{(k!)^n}\sum_{\pi_1,\pi_2,\cdots,\pi_n} \varphi\left(\sum_{i=1}^{[n(\cdot)]} \xi_{ni\pi_i(j)}\right) - \int\limits_{(C[0,1])^d} \varphi(\tilde{W})d\mu \overset{P}{\longrightarrow} 0,$$

where $\pi_1,\pi_1,\cdots,\pi_1$ here ranging over all bijective functions, defined on $\{1,2,\cdots,k\}$. Actually, we shall prove more than that, and definitely, we shall prove its validity but almost everywhere whith respect tp P. For this, let $\omega_1 \in \Omega$ be an arbitrary but temporarily fixed element of those fulfill the hypotheses. We want to prove

$$\frac{1}{(k!)^n}\sum_{\pi_1,\pi_2,\cdots,\pi_n} \varphi\left(\sum_{i=1}^{[n(\cdot)]} \xi_{ni\pi_i(j)}(\omega_1)\right) -$$

$$- \int\limits_{(C[0,1])^d} \varphi\left(\sqrt{\tfrac{n}{k}}\left(\sqrt{s^2(\underline{\xi}_{k_n})}\right)(\omega_1)W + (\cdot)\tfrac{n}{k}\sum_{j=1}^{k}\xi_{n\cdot j}(\omega_1)\right)d\mu \longrightarrow 0,$$

But this is valid indeed from remark 2.3.10.
Now, we turn to prove it in the general case.
Let us define $C_n^M := \left\{ \parallel \frac{n}{k} \sum\limits_{j=1}^{k} \xi_{n\cdot j} \parallel \leqslant M, \parallel \frac{n}{k} s^2(\underline{\xi}_n) \parallel \leqslant M \right\}$, and also for each $n \in \mathbb{N}$, and $j = 1, \cdots, k$, and $i = 1, \cdots, n$, we define the new random arrays $\tilde{\xi}_{nij} := 1_{C_n^M} \cdot \xi_{nij}$. It is clear that the new random arrays satisfy the first case of the proof above.
Consequently, the limit

$$\int\limits_{C_n \times (C[0,1])^d} \varphi(1_{C_n^M} \cdot \tilde{\mathcal{S}}_n) dP \otimes \mu - \int\limits_{C_n \times (C[0,1])^d} \varphi(1_{C_n^M} \cdot \tilde{W}) dP \otimes \mu \underset{n \longrightarrow \infty}{\longrightarrow} 0,$$

is valid for all sequences $(C_n)_{n \in \mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where $C_n \in \mathcal{C}_n$. We can rewrite the previous limit as the following

$$\int\limits_{C_n \times (C[0,1])^d} 1_{C_n^M} \cdot \varphi(\tilde{\mathcal{S}}_n) dP \otimes \mu - \int\limits_{C_n \times (C[0,1])^d} 1_{C_n^M} \cdot \varphi(\tilde{W}) dP \otimes \mu \underset{n \longrightarrow \infty}{\longrightarrow} 0.$$

Also, we can enlarge M to make the following inequality valid $P(C_n^M) \geqslant 1 - \varepsilon$ for any given $\varepsilon > 0$, and for all $n \in \mathbb{N}$.
Therefore, we obtain the validity of the limit

$$\int\limits_{C_n \times (C[0,1])^d} \varphi(\tilde{\mathcal{S}}_n) dP \otimes \mu - \int\limits_{C_n \times (C[0,1])^d} \varphi(\tilde{W}) dP \otimes \mu \underset{n \longrightarrow \infty}{\longrightarrow} 0,$$

for all sequences $(C_n)_{n \in \mathbb{N}}$, and for all bounded uniformly continuous functions $\varphi : (D[0,1])^d \longrightarrow \mathbb{R}$, where $C_n \in \mathcal{C}_n$.
Hence, the proof is complete. $\square$

Corollary 3.4.5. In the same situation of theorem 3.4.4, we have for $j = 1, 2, \cdots, k$, the validity of

$$\tilde{\mathcal{S}}_{nj} \overset{w}{\sim} \tilde{W},$$

or more explicitly

$$\left(\sum_{i=1}^{[nt]} \xi_{nij} \right)_{0 \leqslant t \leqslant 1} \overset{w}{\sim} \left(\sqrt{\frac{n}{k}} \left(\sqrt{s^2(\underline{\xi}_n)} \right) W_t + t \frac{n}{k} \sum_{j=1}^{k} \xi_{n\cdot j} \right)_{0 \leqslant t \leqslant 1}.$$

In the same situation of theorem 3.4.4, we want to discuss the conditional asymptotic normality of a given linear combination of the random arrays $S_{nj}(t_1)$, $S_{nj}(t_2)$,$\cdots$, $S_{nj}(t_l)$, where t_1, t_2,$\cdots$, t_l belong to $[0,1]$, and $j \in \{1,2,\cdots,k\}$. The following remark is for this purpose.

Remark 3.4.6. Let X_{nj} be defined by $\begin{pmatrix} S_{nj}(t_1) \\ \vdots \\ S_{nj}(t_l) \end{pmatrix}$, where t_1, t_2,$\cdots$, t_l belong to $[0,1]$, $j \in \{1,2,\cdots,k\}$, and let a_1, a_2,$\cdots$, a_l be given constant matrices of order $m \times d$, then we have
$P * \sum\limits_{i=1}^{l} a_i \cdot S_{nj}(t_i) \sim \mathcal{N}\left(\mu_n, \sigma_n^2\right)\ (\mathcal{C}_n)$, where $\forall n \in \mathbb{N}$

$$\mu_n := \sum_{i=1}^{l} t_i a_i \cdot \left(\frac{n}{k} \sum_{j=1}^{k} \xi_{n \cdot j} \right)$$

$$\sigma_n^2 := (a_1, \cdots, a_l) \cdot \left(\min(t_i,t_j) \frac{n}{k} s^2 \left(\underline{\xi}_n \right) \right)_{1 \leqslant i,j \leqslant l} \cdot \begin{pmatrix} a_1^t \\ \vdots \\ a_l^t \end{pmatrix}$$

Proof: From Theorem 3.4.4 we conclude the following fact

$$P * \begin{pmatrix} S_{nj}(t_1) \\ \vdots \\ S_{nj}(t_l) \end{pmatrix} \sim \mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right)\ (\mathcal{C}_n),$$

where here $\forall n \in \mathbb{N}$

$$\tilde{\mu}_n := \begin{pmatrix} t_1 \frac{n}{k} \sum\limits_{j=1}^{k} \xi_{n \cdot j} \\ \vdots \\ t_l \frac{n}{k} \sum\limits_{j=1}^{k} \xi_{n \cdot j} \end{pmatrix},$$

$$\tilde{\sigma}_n^2 := \left(\min(t_i,t_j) \frac{n}{k} s^2 (\underline{\xi}_n) \right)_{1 \leqslant i,j \leqslant l}.$$

Which means by definition the validity of the following limit

$$E_P\left(f\left(S_{nj}(t_1), S_{nj}(t_2), \cdots, S_{nj}(t_l)\right) | \mathcal{C}_n\right) - \int f d\mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right) \xrightarrow{P} 0,$$

for all bounded uniformly continuous functions $f : (\mathbb{R}^d)^l \longrightarrow \mathbb{R}$. Now, let $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a bounded uniformly continuous function, and let a_1, a_2,$\cdots$, $a_l \in \mathcal{M}_{m\times d}$ be given constant matrices, then the function $h : (\mathbb{R}^d)^l \longrightarrow \mathbb{R}$, defined by $h(x) := g((a_1, a_2, \cdots, a_l)\cdot x)$, $\forall x \in (\mathbb{R}^d)^l$ is also bounded uniformly continuous function. Thus, we have for this h the validity of the limit $E_P\left(h\left(S_{nj}(t_1), S_{nj}(t_2), \cdots, S_{nj}(t_l)\right)|\mathcal{C}_n\right) - \int h d\mathcal{N}\left(\tilde{\mu}_n, \tilde{\sigma}_n^2\right) \xrightarrow{P} 0$, which can be rewritten as $E_P\left(g\left((a_1, a_2, \cdots, a_l)\cdot\begin{pmatrix} S_{nj}(t_1) \\ \vdots \\ S_{nj}(t_l)\end{pmatrix}\right)\middle|\mathcal{C}_n\right) - \int g d\mathcal{N}\left(\mu_n, \sigma_n^2\right) \xrightarrow{P} 0$, which is also equivalent to

$$E_P\left(g\left(\sum_{i=1}^{l} a_i \cdot S_{nj}(t_i)\right)|\mathcal{C}_n\right) - \int g d\mathcal{N}\left(\mu_n, \sigma_n^2\right) \xrightarrow{P} 0,$$

where here

$$\mu_n := (a_1, a_2, \cdots, a_l)\cdot\begin{pmatrix} t_1 \frac{n}{k}\sum\limits_{j=1}^{k}\xi_{n\cdot j} \\ \vdots \\ t_l \frac{n}{k}\sum\limits_{j=1}^{k}\xi_{n\cdot j}\end{pmatrix} = \sum_{i=1}^{l} t_i a_i \cdot \left(\frac{n}{k}\sum_{j=1}^{k}\xi_{n\cdot j}\right),$$

and

$$\sigma_n^2 := (a_1, \cdots, a_l)\cdot\left(\min(t_i, t_j)\frac{n}{k}s^2\left(\underline{\xi}_n\right)\right)_{1\leqslant i,j\leqslant l}\cdot\begin{pmatrix} a_1^t \\ \vdots \\ a_l^t\end{pmatrix}.$$

Therefore, the proof is complete. □

Corollary 3.4.7. In remark 3.4.6 if $0 < t_1 < t_2 < \cdots < t_l < 1$, then we can rewrite σ_n^2 as the following

$$\sigma_n^2 = \sum_{j=1}^{l}\left(\sum_{i=1}^{j} a_i \frac{n}{k}s^2(\underline{\xi}_n)a_j^t t_i + \sum_{i=1}^{l-j} a_{j+i}\frac{n}{k}s^2(\underline{\xi}_n)a_j^t t_j\right).$$

Also, if $t = \frac{1}{l}$, $a_i = \alpha_i - \alpha_{i+1}$, and $a_l = \alpha_l$, where α_1, α_2,$\cdots$, α_l, are given constant matrices of order $m \times d$, and if $t_i = it$, for $i = 1, 2, \cdots, l$, then we have easily

$\mu_n = t\left(\sum\limits_{i=1}^{l} \alpha_i\right) \cdot \left(\frac{n}{k}\sum\limits_{j=1}^{k} \xi_{n \cdot j}\right)$, and

$\sigma_n^2 = \left(\sum\limits_{i=1}^{l} \alpha_i \frac{n}{k} s^2 \left(\underline{\xi}_n\right) \alpha_i^t\right) t$. These results are involved in the proof of theorem 3.4.9. □

Throughout the proofs of the coming limit theorems in this section we need to represent some Partial sums as path integrals. Let ξ_{n11}, $\xi_{n12}, \cdots, \xi_{n1k}$; $\xi_{n21}, \xi_{n22}, \cdots, \xi_{n2k}; \cdots; \xi_{nn1}, \xi_{nn2}, \cdots, \xi_{nnk}$, be random arrays from $(\Omega, \mathcal{A}, P)$ to $(\mathbb{R}^d, \mathbb{B}^d)$, $S_{nj}(t) := \sum\limits_{i=1}^{[nt]} \xi_{nij}$, and let ω_{ni}, $i = 1, 2, \cdots, n$, $n \in \mathbb{N}$, $j = 1, 2, \cdots, k$, be constant matrices of order $m \times d$. One can write

$$\sum_{i=1}^{n} \omega_{ni} \xi_{nij} = \int_0^1 \omega_n(t) dS'(t),$$

where $S'_{nj}(t)$ is defined by

$$S'_{nj}(t) = S_{nj}\left(\frac{i-1}{n}\right) + n\left(t - \frac{i-1}{n}\right)\left(S_{nj}\left(\frac{i}{n}\right) - S_{nj}\left(\frac{i-1}{n}\right)\right),$$

where $i := [nt] + 1$ in this formula, and $\omega_n \in \mathcal{L}^2([0, 1], \mathcal{M}_{m \times d})$ is defined such that $\omega_{ni} = n \int\limits_{\frac{i-1}{n}}^{\frac{i}{n}} \omega_n(t) dt$, for $i = 1, \cdots, n$, $j = 1, 2, \cdots, k$. Also, we need to see the behavior of $\| \sum\limits_{i=1}^{n} \omega_{ni} \xi_{nij} \|$ when $\| \omega_n \|_2$ tends to zero with n tends to infinity, the next lemma gives an answer to that question.

Lemma 3.4.8. If the random arrays ξ_{n11}, ξ_{n12}, $\cdots$, ξ_{n1k}; ξ_{n21}, ξ_{n22}, $\cdots$, ξ_{n2k}; $\cdots$; ξ_{nn1}, ξ_{nnk} are H_3-exchangeable under $P(\cdot|\mathcal{C}_n)$, and the sequences $\left(\frac{n}{k}\sum\limits_{j=1}^{k} \xi_{n \cdot j}\right)_{n \in \mathbb{N}}$, and $\left(\frac{n}{k} s^2(\underline{\xi}_n)\right)_{n \in \mathbb{N}}$ are stochastically bounded, then

$$\| \sum_{i=1}^{n} \omega_{ni} \xi_{nij} \| \xrightarrow{P} 0, \text{ if } \| \omega_n \|_2 \underset{n \longrightarrow \infty}{\longrightarrow} 0.$$

Proof: Since the sequence $\left(\frac{n}{k}\sum\limits_{i=1}^{n} \xi_{n \cdot j}\right)_{n \in \mathbb{N}}$ is stochastically bounded, it is sufficient to prove

$$\| \sum_{i=1}^{n} \omega_{ni} \left(\xi_{nij} - \bar{\xi}_n\right) \| \xrightarrow{P} 0, \text{ if } \| \omega_n \|_2 \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

where as we know $\bar{\xi}_n := \frac{1}{k} \sum_{j=1}^{k} \xi_{n \cdot j}$.

Since $\omega \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d})$, and from Lemma 3.4.3, we have

$$E\left(\left(\sum_{i=1}^{n} \omega_{ni} \left(\xi_{nij} - \bar{\xi}_n\right)\right)\left(\sum_{i=1}^{n} \omega_{ni} \left(\xi_{nij} - \bar{\xi}_n\right)\right)^t \middle| \mathcal{S}^3\left(\underline{\xi}_n, \mathcal{C}_n\right)\right) =$$

$$= \frac{1}{n} \sum_{i=1}^{n} \omega_{ni} \tfrac{n}{k} s^2(\underline{\xi}_n) \omega_{ni}^t.$$

Now, since

$$\frac{1}{n} \sum_{i=1}^{n} \| \omega_{ni} \|^2 \leqslant \int_0^1 \| \omega_n \|^2 \, dt = \| \omega_n \|_2^2,$$

and since the sequence $\left(\frac{n}{k} s^2(\underline{\xi}_n)\right)_{n\in\mathbb{N}}$ is stochastically bounded, we obtain

$$E\left(\left(\sum_{i=1}^{n} \omega_{ni} \left(\xi_{nij} - \bar{\xi}_n\right)\right)\left(\sum_{i=1}^{n} \omega_{ni} \left(\xi_{nij} - \bar{\xi}_n\right)\right)^t \middle| \mathcal{S}^3\left(\underline{\xi}_n, \mathcal{C}_n\right)\right) \xrightarrow{P} 0.$$

Therefore,

$$E\left(\| \sum_{i=1}^{n} \omega_{ni} \left(\xi_{nij} - \bar{\xi}_n\right) \|^2 \middle| \mathcal{S}^3\left(\underline{\xi}_n, \mathcal{C}_n\right)\right) \xrightarrow{P} 0,$$

and from theorem 1.1.3 we conclude that the assertion is valid indeed. □

In the next theorem we discuss the asymptotic normality of the sum $\sum_{i=1}^{n} \omega_{ni} \xi_{nij}$.

Theorem 3.4.9. Suppose that the triangular array $\underline{\xi}_n$ is satisfying that the sequences $\left(\frac{n}{k} \sum_{j=1}^{k} \xi_{n \cdot j}\right)_{n\in\mathbb{N}}$, and $\left(\frac{n}{k} s^2(\underline{\xi}_n)\right)_{n\in\mathbb{N}}$ are stochastically bounded, and further that sequence $(\omega_n)_{n\in\mathbb{N}}$ is a relatively compact sequence in $\mathcal{L}^2([0,1], \mathcal{M}_{m\times d})$. We call it a sequence of weight functions and assume that these weights are constants. If $\mathcal{C}_n \supseteq \mathcal{S}^3\left(\underline{\xi}_n\right)$, $n \in \mathbb{N}$, is a sequence of sub-σ-fields of $\mathcal{A}$, and if the rows of the triangular array are H_3-exchangeable under $P(\cdot|\mathcal{C}_n)$, then we have

$$P * \sum_{i=1}^{n} \omega_{ni}\xi_{nij} \sim \mathcal{N}(\mu_n, \sigma_n^2)\ (\mathcal{C}_n),$$

where

$$\mu_n := \int_0^1 \omega_n(t)dt \cdot \frac{n}{k}\sum_{j=1}^{k} \xi_{n\cdot j},$$

and

$$\sigma_n^2 := \int_0^1 \omega_n(t)\frac{n}{k}s^2(\underline{\xi}_n)\omega_n^t(t)dt.$$

Proof: Since, $(\omega_n(t))_{n\in\mathbb{N}}$ is relatively compact, so every subsequence of $(\omega_n(t))_{n\in\mathbb{N}}$ contains another convergent sub-subsequence, and consequently it is sufficient to prove the assertion in the case when the sequence $(\omega_n(t))_{n\in\mathbb{N}}$ is convergent. And from lemma 3.4.8 we conclude that it is sufficient to prove it in the case $\omega_n(t) = \omega(t)$, $\forall n \in \mathbb{N}$. Moreover, remarking that any $\omega \in \mathcal{L}^2([0,1], \mathcal{M}_{m\times d})$ can assumed as a limit of some sequence of step-functions belong to $M_{m\times d}$, where $M_{m\times d} := \left\{\omega : \omega(t) = \sum_{i=1}^{l} x_i \cdot 1_{[\frac{i-1}{l},\frac{i}{l})}(t), x_i \in \mathcal{M}_{m\times d},\ i = 1,\cdots,l,\ l \in \mathbb{N}\right\}$. By applying lemma 3.4.8 we find that it is sufficient to prove it when $\omega \in M_{m\times d}$. Now, we note that $\sum_{i=1}^{n} \omega_{ni}\xi_{nij} = \int_0^1 \omega(t)dS'_{nj}(t) = \sum_{i=1}^{l} x_i \left(S'_{nj}\left(\frac{i}{l}\right) - S'_{nj}\left(\frac{i-1}{l}\right)\right) = \sum_{i=1}^{l} a_i S'_{nj}\left(\frac{i}{l}\right)$, where $a_i := x_i - x_{i+1}$, $i = 1, 2, \cdots, l-1$, $a_l = x_l$. Since the triangular array $\underline{\xi}_n$ is infintesimal, and from remark 3.4.6, and corollary 3.4.7, we find that the assertion is valid indeed. □

Remark 3.4.10. In theorem 3.4.9, σ_n^2 can be rewritten as $\sigma_n^2 = \frac{1}{n}\sum_{i=1}^{n} \omega_{ni}\frac{n}{k}s^2(\underline{\xi}_n)\omega_{ni}^t$, $n \in \mathbb{N}$. □

We want at this position to introduce some important properties of the concepts of the H_3-exchangeablity under $P(\cdot|\mathcal{C})$, where $\mathcal{C}$ is a given sub-σ-field of $\mathcal{A}$, also we want to discuss some properties of the σ-field $\mathcal{S}^3\left(\underline{\xi}_n\right)$, $n \in \mathbb{N}$. For this, we shall introduce the definition of the H_3-exchangeablity under P, also some interesting results.

Definition 3.4.11. Under the same hypotheses of definition 3.4.1, we call the triangular array $\underline{\xi}_n$ H_3-exchangeable under P iff this triangular array is H_3-exchangeable under $P(\cdot|\mathcal{C})$, for $\mathcal{C} = \{\emptyset, \Omega\}$.

– It is clear from lemma 3.4.3 that the H_3-exchangeablity under $P(\cdot|\mathcal{C})$, for all $\mathcal{C} \subseteq \mathcal{S}^3\left(\underline{\xi}_n\right)$ is equivalent to the H_3-exchangeablity under P.

Lemma 3.4.12. Let X_{11}, X_{12}, $\cdots$, X_{1k}; X_{21}, X_{22}, $\cdots$, X_{2k}; $\cdots$; X_{n1}, X_{n2}, $\cdots$, X_{nk}; be random elements from $(\Omega, \mathcal{A}, P)$ to $(\mathcal{E}, \mathcal{B})$, where $\mathcal{E}$ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra defined on $\mathcal{E}$. Let further, $f : \mathcal{E} \times \mathcal{E}^{kn} \longrightarrow \mathbb{R}^m$ be a measurable function, which is permutation symmetric in its second variable, and let $\xi_{ij} := f\left(X_{ij}, \underline{X}_n\right)$, $i = 1, \cdots, n$, $j = 1, \cdots, k$, then $\mathcal{S}^3\left(\underline{\xi}_n\right) \subseteq \mathcal{S}^3\left(\underline{X}_n\right)$.

Proof: It is sufficient to prove that the following implication

$$g\left(\underline{\xi}_n\right) \text{ is } \mathcal{S}^3\left(\underline{\xi}_n\right)\text{-measurable} \Longrightarrow g\left(\underline{\xi}_n\right) \text{ is } \mathcal{S}^3\left(\underline{X}_n\right)\text{-measurable}$$

is valid for all measurable $g : (\mathbb{R}^m)^{kn} \longrightarrow \mathbb{R}$. For this, let $h : \mathcal{E}^{kn} \longrightarrow (\mathbb{R}^m)^{kn}$ be defined by

$$h(\underline{x}_n) := \left(f\left(x_{11}, \underline{x}_n\right), \cdots, f\left(x_{1k}, \underline{x}_n\right); \cdots; f\left(x_{n1}, \underline{x}_n\right), \cdots, f\left(x_{nk}, \underline{x}_n\right)\right),$$

where $\underline{x}_n = (x_{11}, \cdots, x_{1k}; \cdots; x_{n1}, \cdots, x_{nk}) \in \mathcal{E}^{kn}$. Let $g\left(\underline{\xi}_n\right)$ be $\mathcal{S}^3\left(\underline{\xi}_n\right)$-measurable, this implies $g\left(\theta\left(\underline{\xi}_n\right)\right) = g\left(\underline{\xi}_n\right)$ for all $\theta \in \Theta_n$. Let us define $g_0 : \mathcal{E}^{kn} \longrightarrow \mathbb{R}$ by $g_0\left(\underline{x}_n\right) := g(h(\underline{x}_n))$.
Hence, $g_0\left(\underline{X}_n\right) = g\left(h\left(\underline{X}_n\right)\right) = g\left(\underline{\xi}_n\right)$, and $g_0\left(\theta\left(\underline{X}_n\right)\right) = g\left(h\left(\theta\left(\underline{X}_n\right)\right)\right) = g\left(\theta\left(\underline{\xi}_n\right)\right) = g\left(\underline{\xi}_n\right) = g_0\left(\underline{X}_n\right)$.
Hence, $g_0\left(\theta\left(\underline{X}_n\right)\right) = g_0\left(\underline{X}_n\right)$, but this means that $g_0\left(\underline{X}_n\right)$ is $\mathcal{S}^3\left(\underline{X}_n\right)$-measurable, and also $g\left(\underline{\xi}_n\right)$ is $\mathcal{S}^3\left(\underline{X}_n\right)$-measurable. Consequently, the assertion is valid indeed. □

Lemma 3.4.13. Let $f : \mathcal{E} \times \mathcal{E}^{kn} \longrightarrow \mathbb{R}^m$ be a measurable function which is permutation symmetric in its second variable. If the random elements X_{11}, $\cdots$, X_{1k}; $\cdots$; X_{n1}, $\cdots$, X_{nk} are H_3-exchangeable under $P(\cdot|\mathcal{C})$, then $f(X_{11}, \underline{X}_n), \cdots, f(X_{1k}, \underline{X}_n); \cdots; f(X_{n1}, \underline{X}_n), \cdots, f(X_{nk}, \underline{X}_n)$ are also H_3-exchangeable under $P(\cdot|\mathcal{C})$.

Proof: It is sufficient to prove that for any measurable function $g : (\mathbb{R}^m, \mathbb{B}^m)^{kn} \longrightarrow (\mathbb{R}, \mathbb{B})$ satisfying that the quantity

$$E\left(g\left(f\left(X_{11}, \underline{X}_n\right), \cdots, f\left(X_{1k}, \underline{X}_n\right); \cdots; f\left(X_{n1}, \underline{X}_n\right), \cdots, f\left(X_{nk}, \underline{X}_n\right)\right)\right)$$

is well-defined, the equality

$$E\left(g\left(\theta\left(f\left(X_{11}, \underline{X}_n\right), \cdots, f\left(X_{1k}, \underline{X}_n\right); \cdots; f\left(X_{n1}, \underline{X}_n\right), \cdots, f\left(X_{nk}, \underline{X}_n\right)\right)\right)|\mathcal{C}\right) =$$
$$= E\left(g\left(f\left(X_{11}, \underline{X}_n\right), \cdots, f\left(X_{1k}, \underline{X}_n\right); \cdots; f\left(X_{n1}, \underline{X}_n\right), \cdots, f\left(X_{nk}, \underline{X}_n\right)\right)|\mathcal{C}\right)$$

is valid almost sure with respect to P. For this, let us define $h : \mathcal{E}^{kn} \longrightarrow (\mathbb{R}^m)^{kn}$ by
$h(\underline{x}_n) = \left(f\left(x_{11}, \underline{x}_n\right), \cdots, f\left(x_{1k}, \underline{x}_n\right); \cdots; f\left(x_{n1}, \underline{x}_n\right), \cdots, f\left(x_{nk}, \underline{x}_n\right)\right)^t$, where $\underline{x}_k = (x_{11}, \cdots, x_{1k}; \cdots; x_{n1}, \cdots, x_{nk}) \in \mathcal{E}^{kn}$. And we can write
$h(\underline{X}_n) = \left(f\left(X_{11}, \underline{X}_n\right), \cdots, f\left(X_{1k}, \underline{X}_n\right); \cdots; f\left(X_{n1}, \underline{X}_n\right), \cdots, f\left(X_{nk}, \underline{X}_n\right)\right)^t$.
Consequently, it remains to be proved the validity of the equality

$$E\left(g\left(\theta\left(h\left(\underline{X}_n\right)\right)\right)|\mathcal{C}\right) = E\left(g\left(h\left(\underline{X}_n\right)\right)|\mathcal{C}\right) \ [P].$$

But easily, we see that the left side is equal to $E\left(g\left(h\left(\theta\left(\underline{X}_n\right)\right)\right)|\mathcal{C}\right)$ almost sure under P. Since $X_{11}, \cdots, X_{1k}; \cdots; X_{n1}, \cdots, X_{nk}$ are H_3-exchangeable under $P(\cdot|\mathcal{C})$, the assertion follows. □

Lemma 3.4.14. Let $\xi_{11}, \cdots, \xi_{1k}; \cdots; \xi_{n1}, \cdots, \xi_{nk} : (\Omega, \mathcal{A}, P) \longrightarrow (\mathbb{R}^m, \mathbb{B}^m)$ be random arrays statisfying that $\sum_{j=1}^{k} \xi_{ij}$, and $\sum_{j=1}^{k} \xi_{ij}\xi_{ij}^t$ are independent of i and j. Let further $\mathcal{C} \subseteq \mathcal{A}$ be a sub-σ-field. Let $\underline{x}_n = (x_{11}, \cdots, x_{1k}, \cdots, x_{n1}, \cdots, x_{nk}) \in S \subseteq (\mathbb{R}^m)^{kn}$, where $S := \{\underline{x}_n : \sum_{j=1}^{k} x_{ij}$ and $\sum_{j=1}^{k} x_{ij}x_{ij}^t$ are independent of i, j $\}$, and $\bar{x}_n = \frac{1}{k}\sum_{j=1}^{k} x_{\cdot j}$, and $V(\underline{x}_n) = \sum_{j=1}^{k}(x_{\cdot j} - \bar{x}_n)(x_{\cdot j} - \bar{x}_n)^t$. Assume that $V(\underline{x}_n)$ is positive definite, and let $g : \mathbb{R}^m \times S \longrightarrow \mathbb{R}^m$ be defined by $g(y, \underline{x}_n) = \left(V\left(\underline{x}_n\right)\right)^{-\frac{1}{2}}(y - \bar{x}_n)$, and $\eta_{ij} = g\left(\xi_{ij}, \underline{\xi}_n\right)$, $i = 1, \cdots, n$, $j = 1, \cdots, k$. If $\xi_{11}, \cdots, \xi_{1k}; \cdots; \xi_{n1}, \cdots, \xi_{nk}$ are H_3-exchangeable under $P(\cdot|\mathcal{C})$, and if $\bar{\xi}_n$ and $V(\underline{\xi}_n)$ are $\mathcal{C}$-measurable, then $\mathcal{S}^3\left(\underline{\xi}_n, \mathcal{C}\right) = \mathcal{S}^3\left(\underline{\eta}_n, \mathcal{C}\right)$.

Proof: First we see that $\mathcal{S}^3\left(\underline{\eta}_n, \mathcal{C}\right) \subseteq \mathcal{S}^3\left(\underline{\xi}_n, \mathcal{C}\right)$ follows from lemma 3.4.12 because g is permutation symmetric in its second variable. It remains to show that $\mathcal{S}^3\left(\underline{\xi}_n, \mathcal{C}\right) \subseteq \mathcal{S}^3\left(\underline{\eta}_n, \mathcal{C}\right)$. And for this it is sufficient to prove that $\mathcal{S}^3\left(\underline{\xi}_n\right) \subseteq \mathcal{S}^3\left(\underline{\eta}_n, \mathcal{C}\right)$. Let $f : S \longrightarrow \mathbb{R}$ be any measurable function such that $f\left(\underline{\xi}_n\right)$ is $\mathcal{S}^3\left(\underline{\xi}_n\right)$-measurable. This means $f\left(\theta\left(\underline{\xi}_n\right)\right) = f\left(\underline{\xi}_n\right)$ for all $\theta \in \Theta_n$.
To complete this proof we have to prove that $f\left(\underline{\xi}_n\right)$ is $\mathcal{S}^3\left(\underline{\eta}_n, \mathcal{C}\right)$-measurable. For this, let $z = (z_{11}, \cdots, z_{nk}) \in S$, $a \in \mathbb{R}^m$, $B \in \mathcal{M}_{m\times m}$, and B is positive definite and symmetric, h be defined by $h(z, a, B) = (B^{\frac{1}{2}} z_{11} + a, \cdots, B^{\frac{1}{2}} z_{nk} + a)^t$. It is easy to see that $h(\theta(z), a, B) = \theta(h(z, a, B))$, $\underline{\xi}_n = h(\underline{\eta}_n, \bar{\xi}_n, V(\underline{\xi}_n))$, and $f\left(\underline{\xi}_n\right) = f\left(h\left(\underline{\eta}_n, \bar{\xi}_n, V(\underline{\xi}_n)\right)\right)$ but $\bar{\xi}_n$ and $V\left(\underline{\xi}_n\right)$ are $\mathcal{S}^3\left(\underline{\eta}_n, \mathcal{C}\right)$-measurable because $\mathcal{C} \subseteq \mathcal{S}^3\left(\underline{\eta}_n, \mathcal{C}\right)$, and they are $\mathcal{C}$-measurable. But also, $f\left(h\left(\theta\left(\underline{\eta}_n\right), \bar{\xi}_n, V(\underline{\xi}_n)\right)\right) = f\left(h\left(\underline{\eta}_n, \bar{\xi}_n, V(\underline{\xi}_n)\right)\right)$ for all $\theta \in \Theta_n$. Thus, $f\left(\underline{\xi}_n\right) = f\left(h\left(\underline{\eta}_n, \bar{\xi}_n, V(\underline{\xi}_n)\right)\right)$ is $\mathcal{S}^3\left(\underline{\eta}_n, \mathcal{C}\right)$-measurable. □

In the introduction of this section we have defined the statistics $T_{nj} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \omega_{ni} f_n\left(X_{nij}, \underline{X}_n\right)$, for $j = 1, \cdots, k$, we want here to make some preliminary computations to prepare for the next theorem which will discuss the conditional asymptotic normality of the sequence $\left(T_{nj}\right)_{n\in\mathbb{N}}$. For this, assume the same situation of the introduction, and let us first prove that $E\left(f_n\left(X_{nij}, \underline{X}_n\right) | \mathcal{S}^3\left(\underline{X}_n\right)\right)$, and $Var\left(f_n\left(X_{nij}, \underline{X}_n\right) | \mathcal{S}^3\left(\underline{X}_n\right)\right)$, are independent of i and j. One can easily see

$$E\left(f_n\left(X_{nij}, \underline{X}_n\right) | \mathcal{S}^3\left(\underline{X}_n\right)\right) = \frac{1}{k} \sum_{j=1}^{k} f_n\left(X_{nij}, \underline{X}_n\right) \;\; [\mathbb{P}],$$

$$\begin{aligned} &Var\left(f_n\left(X_{nij}, \underline{X}_n\right) | \mathcal{S}^3\left(\underline{X}_n\right)\right) = \\ &= \tfrac{1}{k} \sum_{j=1}^{k} \left(f_n\left(X_{nij}, \underline{X}_n\right) - E\left(f_n\left(X_{nij}, \underline{X}_n\right) | \mathcal{S}^3\left(\underline{X}_n\right)\right)\right) \cdot \\ &\qquad \cdot \left(f_n\left(X_{nij}, \underline{X}_n\right) - E\left(f_n\left(X_{nij}, \underline{X}_n\right) | \mathcal{S}^3\left(\underline{X}_n\right)\right)\right)^t \;\; [\mathbb{P}]. \end{aligned}$$

Therefore, for notational convenience we denote $E\left(f_n\left(X_{nij}, \underline{X}_n\right) | \mathcal{S}^3\left(\underline{X}_n\right)\right)$ by $E\left(f_n | \mathcal{S}^3\left(\underline{X}_n\right)\right)$, i.e.

$$E\left(f_n | \mathcal{S}^3\left(\underline{X}_n\right)\right) := E\left(f_n\left(X_{nij}, \underline{X}_n\right) | \mathcal{S}^3\left(\underline{X}_n\right)\right),$$

and $Var\left(f_n\left(X_{nij},\underline{X}_n\right)|\mathcal{S}^3\left(\underline{X}_n\right)\right)$ by $Var\left(f_n|\mathcal{S}^3\left(\underline{X}_n\right)\right)$, i.e.

$$Var\left(f_n|\mathcal{S}^3\left(\underline{X}_n\right)\right) := Var\left(f_n\left(X_{nij},\underline{X}_n\right)|\mathcal{S}^3\left(\underline{X}_n\right)\right).$$

Let us now give explicit expressions for $E\left(T_{nj}|\mathcal{S}^3\left(\underline{X}_n\right)\right)$, and $Var\left(T_{nj}|\mathcal{S}^3\left(\underline{X}_n\right)\right)$.

$$E\left(T_{nj}|\mathcal{S}^3\left(\underline{X}_n\right)\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\omega_{ni}E\left(f_n|\mathcal{S}^3\left(\underline{X}_n\right)\right)\ [\mathbb{P}],$$

$$Var\left(T_{nj}|\mathcal{S}^3\left(\underline{X}_n\right)\right) = Var\left(\tfrac{1}{\sqrt{n}}\sum_{i=1}^{n}\omega_{ni}f_n\left(X_{nij},\underline{X}_n\right)|\mathcal{S}^3\left(\underline{X}_n\right)\right) =$$
$$= \tfrac{1}{n}\sum_{i=1}^{n}\omega_{ni}Var\left(f_n|\mathcal{S}^3\left(\underline{X}_n\right)\right)\omega_{ni}^t +$$
$$+\tfrac{1}{n}\sum_{1\leqslant i\neq l\leqslant n}\omega_{ni}E\Big(\left(f_n\left(X_{nij},\underline{X}_n\right)-E\left(f_n|\mathcal{S}^3\left(\underline{X}_n\right)\right)\right)\cdot$$
$$\cdot\left(f_n\left(X_{nlj},\underline{X}_n\right)-E\left(f_n|\mathcal{S}^3\left(\underline{X}_n\right)\right)\right)^t|\mathcal{S}^3\left(\underline{X}_n\right)\Big)\omega_{nl}^t\ [\mathbb{P}].$$

Hence,
$$Var\left(T_{nj}|\mathcal{S}^3\left(\underline{X}_n\right)\right) = \tfrac{1}{n}\sum_{i=1}^{n}\omega_{ni}Var\left(f_n|\mathcal{S}^3\left(\underline{X}_n\right)\right)\omega_{ni}^t\ [\mathbb{P}].$$

– For the next theorem we assume first the same situation in the introduction of this section.

Theorem 3.4.15. Let the triangular array $\underline{\xi}_n$ be defined by

$$\frac{1}{\sqrt{n}}\left(f_n\left(X_{nij},\underline{X}_n\right)-E\left(f_n|\mathcal{S}^3\left(\underline{X}_n\right)\right)\right),\ 1\leqslant i\leqslant n,\ n\in\mathbb{N},\ 1\leqslant j\leqslant k,$$

and assume that the sequence $\left(Var\left(f_n|\mathcal{S}^3\left(\underline{X}_n\right)\right)\right)_{n\in\mathbb{N}}$ is stochastically bounded. Assume further that the sequence $\left(\omega_n(t)\right)_{n\in\mathbb{N}}\subseteq\mathcal{L}^2\left([0,1],\mathcal{M}_{m\times d}\right)$ is relatively compact sequence of non-random weight functions. Then the sequence $\left(T_n\right)_{n\in\mathbb{N}}$ is asymptotically normal under $\mathbb{P}\in H_3$, and conditioned by the symmetric sub-σ-field $\mathcal{S}^3\left(\underline{X}_n\right)$. More explicitly,

$$\mathbb{P}*\left(T_{nj}-E\left(T_{nj}|\mathcal{S}^3\left(\underline{X}_n\right)\right)\right)\sim\mathcal{N}\left(0,Var\left(T_{nj}|\mathcal{S}^3\left(\underline{X}_n\right)\right)\right)\ \left(\mathcal{S}^3\left(\underline{X}_n\right)\right).$$

Proof: Let ξ_{nij}, $i=1,2,\cdots,n$, $n\in\mathbb{N}$, $j=1,2,\cdots,k$, be defined by

$$\xi_{nij} := \frac{1}{\sqrt{n}}\left(f_n\left(X_{nij},\underline{X}_n\right)-E\left(f_n|\mathcal{S}^3\left(\underline{X}_n\right)\right)\right).$$

From lemma 3.4.13, the random arrays $\xi_{n11}, \cdots, \xi_{nk}$, are H_3-exchangeable under $\mathbb{P}\left(\cdot, \mathcal{S}^3\left(\underline{X}_n\right)\right)$. Let

$$F_{nj}(t) := \sum_{i=1}^{[nt]} \frac{1}{\sqrt{n}} \left(f_n \left(X_{nij}, \underline{X}_n \right) - E\left(f_n | \mathcal{S}^3\left(\underline{X}_n\right)\right)\right),$$

$0 \leqslant t \leqslant 1$, and let

$$F'_{nj}(t) := F_{nj}\left(\frac{i-1}{n}\right) + n\left(t - \frac{i-1}{n}\right)\left(F_{nj}\left(\frac{i}{n}\right) - F_{nj}\left(\frac{i-1}{n}\right)\right),$$

where $i := [nt] + 1$.
Therefore, we can write

$$T_{nj} - E\left(T_{nj} | \mathcal{S}^3\left(\underline{X}_n\right)\right) = \int_0^1 \omega_n(t) dF'_{nj}(t).$$

Hence, all assumption of theorem 3.4.9 are satisfied, and consequently we have

$$\mathbb{P} * \left(T_{nj} - E\left(T_{nj} | \mathcal{S}^3\left(\underline{X}_n\right)\right)\right) \sim \mathcal{N}\left(0, \sigma_n^2\right) \ \left(\mathcal{S}^3\left(\underline{X}_n\right)\right).$$

where $\sigma_n^2 := \int_0^1 \omega_n(t) \frac{n}{k} s^2\left(\underline{\xi}_n\right) \omega_n^t(t) dt =$
$= \int_0^1 \omega_n(t) Var\left(f_n | \mathcal{S}^3\left(\underline{X}_n\right)\right) \omega_n^t(t) dt \ [\mathbb{P}]$. Also, from remark 3.4.10 we can write
$\sigma_n^2 = \frac{1}{n} \sum_{i=1}^{n} \omega_{ni} \frac{n}{k} s^2(\underline{\xi}_n) \omega_{ni}^t = \frac{1}{n} \sum_{i=1}^{n} \omega_{ni} Var\left(f_n | \mathcal{S}^3\left(\underline{X}_n\right)\right) \omega_{ni}^t \ [\mathbb{P}]$.
By remarking that $\sigma_n^2 - Var\left(T_{nj} | \mathcal{S}^3\left(\underline{X}_n\right)\right) \xrightarrow{\mathbb{P}} 0$, we conclude that the assertion is valid. $\square$

Remark 3.4.16. From theorem 1.3.4, it is easy to see that for any $j \in \{ 1, \cdots, k \}$ the assertion of theorem 3.4.15 is equivalent to

$$\mathbb{P} * T_{nj} \sim \mathcal{N}\left(E\left(T_{nj} | \mathcal{S}^3\left(\underline{X}_n\right)\right), Var\left(T_{nj} | \mathcal{S}^3\left(\underline{X}_n\right)\right)\right) \ \left(\mathcal{S}^3\left(\underline{X}_n\right)\right),$$

if the sequence $\left(E\left(T_{nj} | \mathcal{S}^3\left(\underline{X}_n\right)\right)\right)_{n \in \mathbb{N}}$ is stochastically bounded.

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Lebenslauf

Name	Al-Hassbani
Vornamen	Basel
Geschlecht	Männlich
Geb.-Datum	06. Januar 1976
Geb.-Ort	Daraa, Syrien
Familienstand	Ledig
Staatsbürgerschaft	Syrisch

Schulbildung und Studium

1981 - 1993	Grundschule und Gymnasium, Damaskus
1993 - 1997	Universität Damaskus Studium der reine Mathematik
1997 - 1998	Diplom: Analyse
1998 - 2000	Assistent an der Universität Damaskus
2000 - 2002	Universität Hamburg Promotionsstudium der Statistik Diplom: Statistik
2002 - 2005	Universität Hamburg Bereitung der Doktorarbeit

Sprachkenntnisse	Arabisch, Englisch, Deutsch
EDV-Kenntnisse	Programmierung
Hobbys	Schach, Sport

Zusammenfassung:

In dieser Dissertation wird eine Reihe von Resultaten im Bereich der Theorie der multivariaten Permutationstests hergeleitet und zwar im Hinblick auf die bedingte asymptotische Normalität verschiedener Permutationsstatistiken, wobei unter gewissen symmetrischen σ-Algebren bedingt wird. Zur Herleitung dieser Resultate mussten viele klassische Aussagen und Theoreme aus der Theorie der stochastischen Prozesse verallgemeinert werden, insbesondere auf den Fall multivariater Beobachtungen. Diese Dissertation hat drei Kapitel. Das erste Kapitel gibt eine Darstellung vieler grundlegender Vorbedingungen über bedingte Erwartungen, die bedingte schwache asymptotische Gleichheit, die bedingte asymptotische Normalität und die Austauschbarkeit. Die Konzepte und damit in Verbindung stehende Theoreme und Beweise wurden in Parallelität zu ähnlichen Fällen in der Literatur entwickelt, z.B. das Buch von Billingsley [1968]: Convergence of Probability Measures, und auch die Arbeit von H. Strasser und C. Weber [1998]: On The Asymptotic Theory of Multivariate Permutation Statistics. Im zweiten Kapitel werden zentrale Grenzwertsätze für stochastische Prozessen bewiesen, aus denen im dritten Kapitel bedingte Grenzwertsätze für Teststatistiken der bekannten Hypothesen H_0, H_1, H_2, und H_3 hergeleitet werden. Entsprechend enthält dieses Kapitel vier Abschnitte, in denen die vier Hypothesen einzeln behandelt werden. Insgesamt gibt diese Dissertation einen Anfang und Ausblick auf eine neue Theorie der Prüfung statistischer Hypothesen, in denen die bedingten Erwartungen die Rolle der Integrale in der vertrauten klassischen Theorie der Prüfung statistischer Hypothesen spielen.